# FE Review Dynamics Notes © 2018


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# Fundamentals of Engineering Review for Dynamics

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**&**

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*Review Notes available in PDF format @*

**http://www.lsu.edu/eng/docs/FE-Exam-Review/Dynamics.pdf**

*FE Exam Reference Handbook (free download)*

**https://account.ncees.org/reference-handbooks/**

*Last Revised: 01/18*

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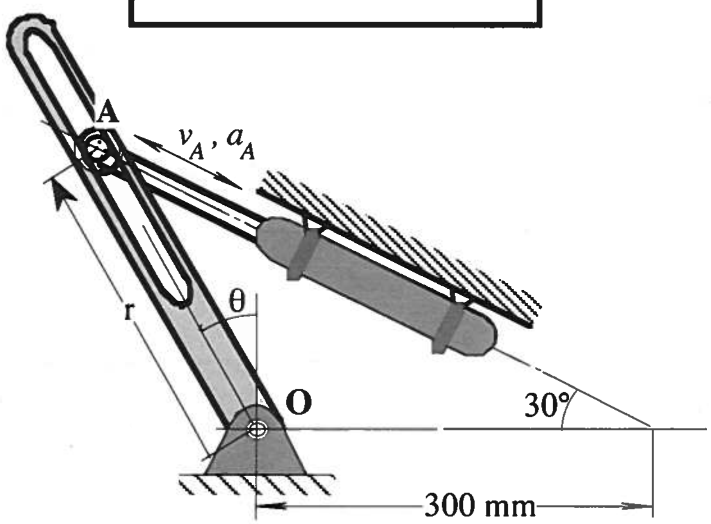
**Louisiana State University**

# Dynamics Problem Decomposition

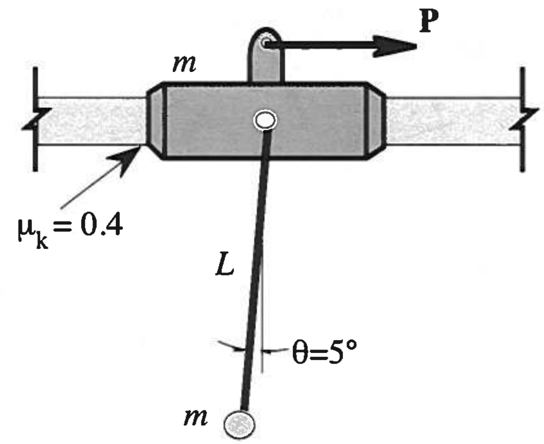
Dynamics

Kinematics

Kinetics




Geometric descriptions of motion & constraints



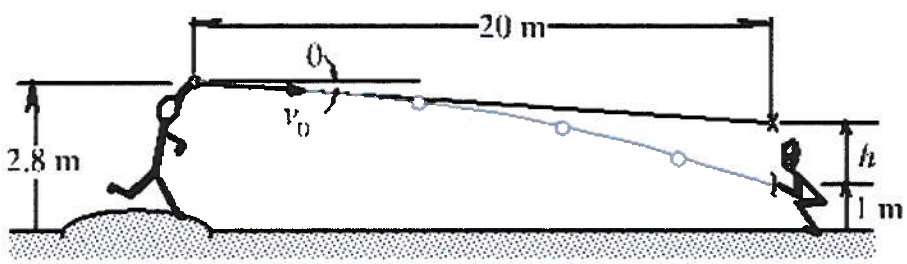

Loading relationships which dictate CHANGES in motion

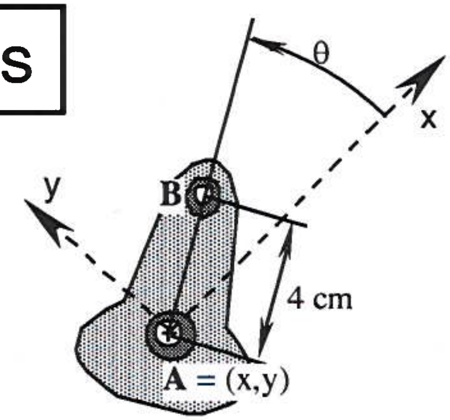
# Dynamic Studies

Plane motion:
DOF (Degrees of Freedom)?

**Dynamics:**
**(Kinematics & Kinetics)**

| Particles | Rigid Bodies |
|---|---|




**Particles**

- $m$ – mass (inertia)
- $\underline{P}$ - position $\{(x,y),(r,\theta)\}$ (2DOF)

  $\underline{V}$ – velocity $\{(v_x, v_y),(v_r, v_\theta),(0,v)\}$

  $\underline{A}$ – acceleration
  $\{(a_x, a_y),(a_r, a_\theta),(a_n, a_t)\}$

**Rigid Bodies**

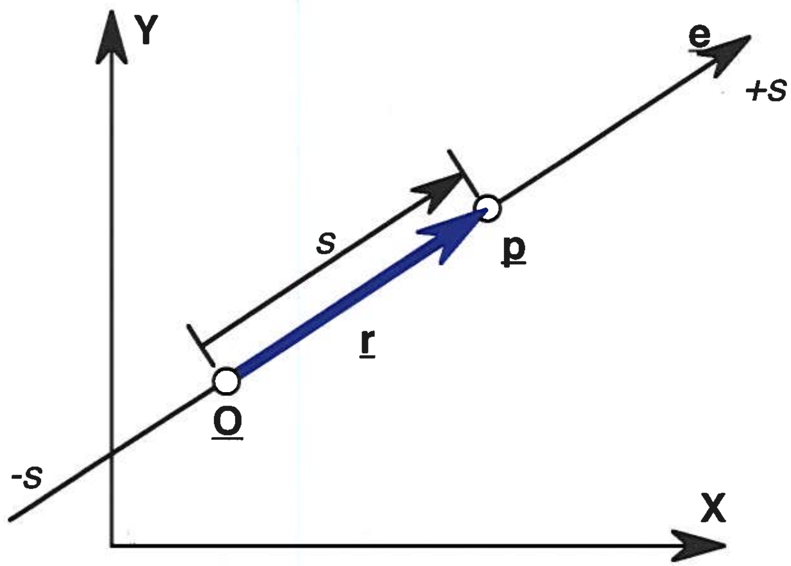
- $m$ & $I$ – add in rotational inertia ($I$)
- $(\underline{P},\theta)$ - position & orientation (3DOF)

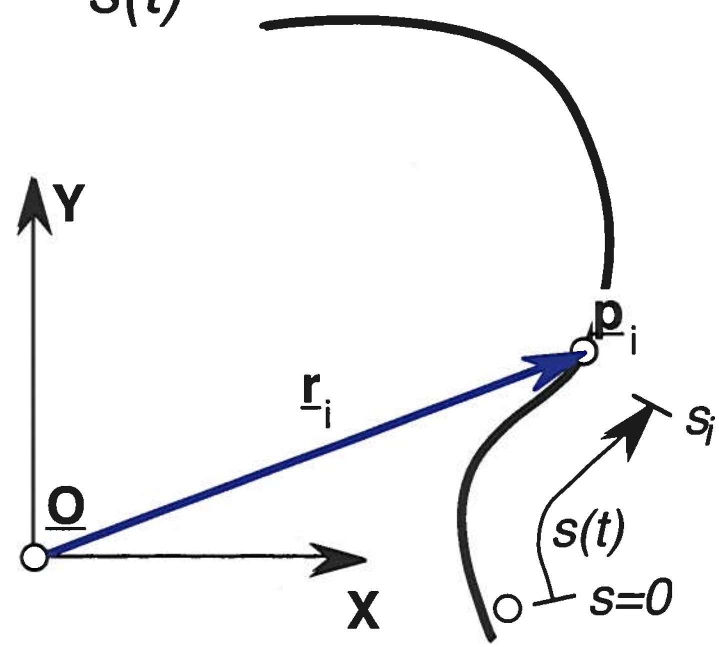
  $\underline{V}$ – velocity $\{\underline{V}_A, \omega\}$

  $\underline{A}$ – acceleration $\{\underline{A}_A, \alpha\}$

## Getting Started => Particle Kinematics

- Rectilinear Motion
  - Movement along a straight line in 1-2 or 3D
    - 1 Degree of Freedom (DOF)* - *s(t)*




- Curvilinear Motion
  - Movement of particle along an arbitrary path through space

# Rectilinear Motion Overview (Calculus/Physics Review!):

D I F F E R E N T I A T E

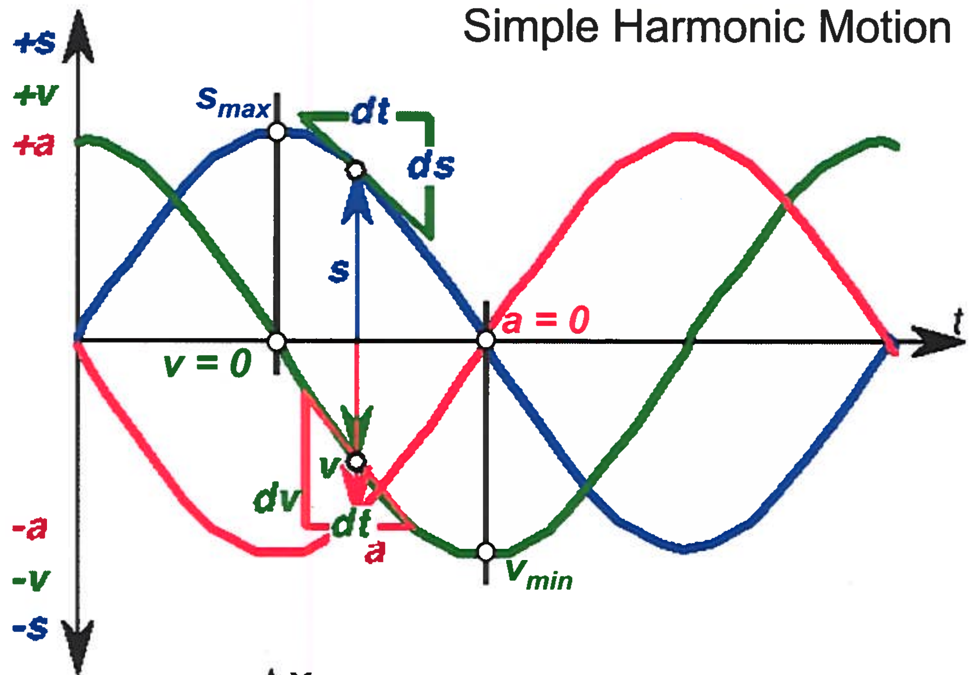
I N T E G R A T E

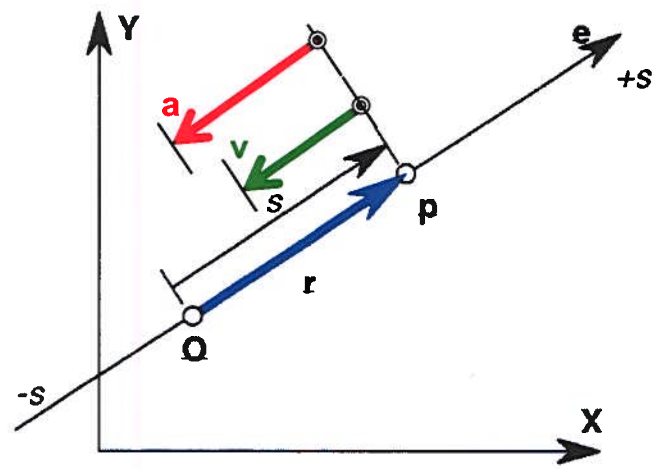
- Position - $s(t)$
- Speed - $v(t)$

$$(1) \quad v = \frac{ds}{dt} = \dot{s}$$

- Acceleration - $a(t)$

$$(2) \quad a = \frac{dv}{dt} = \dot{v} = \frac{d^2 s}{dt^2} = \ddot{s}$$



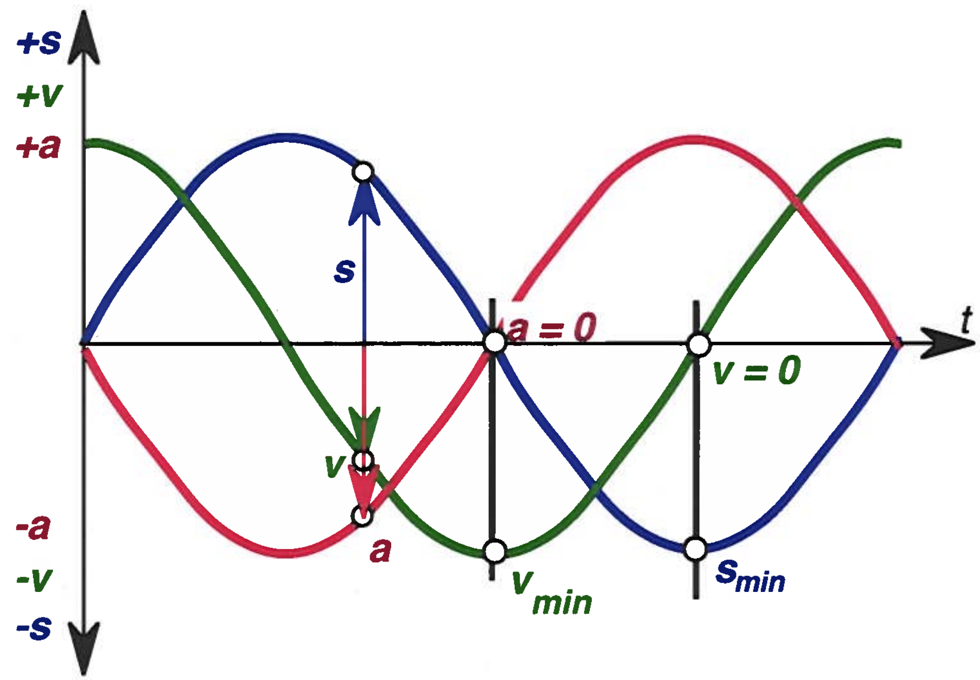

- *Typical Functions ??*
  - Polynomial, Trigonometric, Logarithmic, Exponential

# Rectilinear Motion Summary:

DIFFERENTIATE ↓ INTEGRATE ↑

- Position - $s(t)$
- Speed - $v(t)$

$$(1) \quad v = \frac{ds}{dt} = \dot{s}$$




- Acceleration - $a(t)$

$$(2) \quad a = \frac{dv}{dt} = \dot{v} = \frac{d^2 s}{dt^2} = \ddot{s}$$

- Alternate form ?

$$(2^*) \quad v \; dv = a \; ds$$

---

$a(t)$ => Solid Rocket Propulsion   $a(v)$ => aerodynamic drag

$a(s)$, $v(s)$ => Gravitational fields, springs, conservative forces etc.

---

($s,v,a$ & $t$) => $t$ independent parameter

**Given:** $s = 2t^2 - 8t + 3$

**Find:** **Displacement from t = 1 to t = 3**

**Distance traveled from t = 1 to t = 3**

$s = 2t^2 - 8t + 3$ $\quad s(1) = -3 \quad \& \quad s(3) = -3$ $\quad s(2) = -5$

$v = \dot{s} = 4t - 8$ $\quad v(2) = 0$

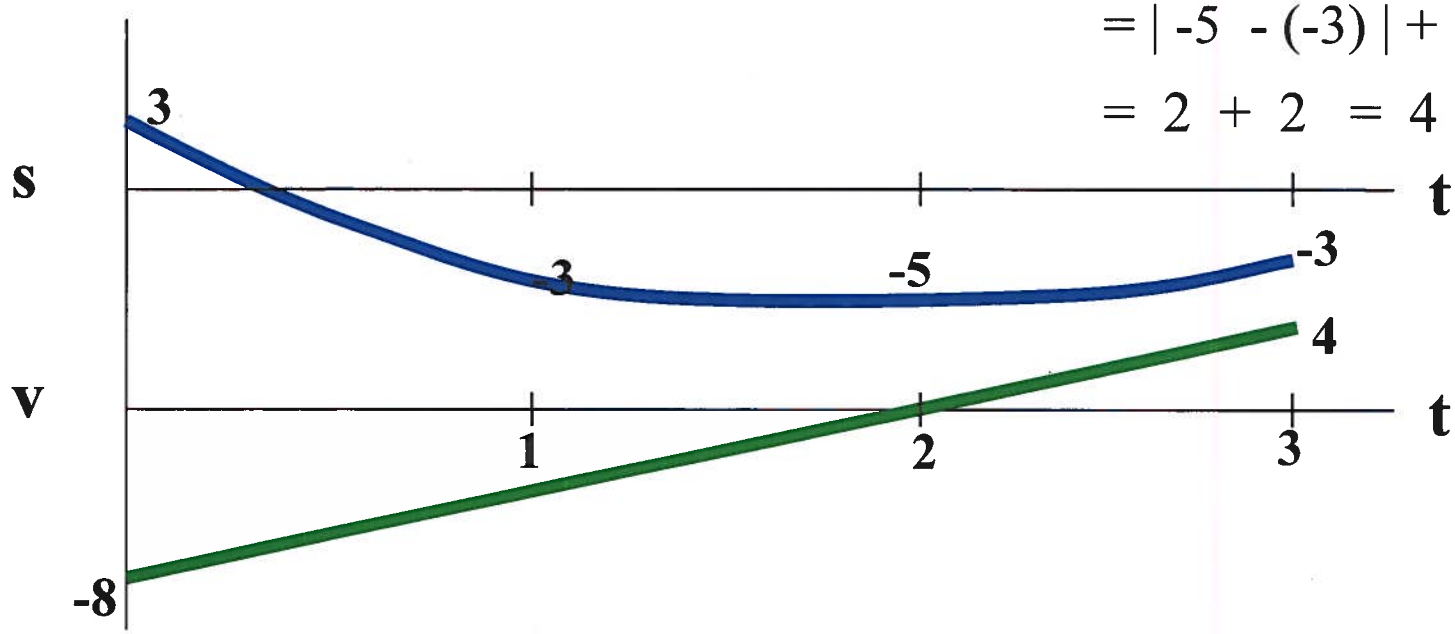
$a = \ddot{s} = 4$ $\quad a(2) = 4$

Since, $s(3) - s(1) = 0$ → displacement = 0

Reversal @ $v|_{t=2} = 0$ distance traveled $= |s(2) - s(1)| + |s(3) - s(2)|$

$= |-5 - (-3)| + |(-3) - (-5)|$

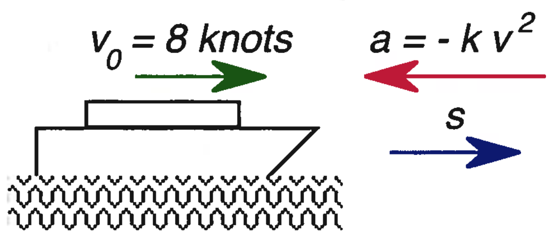
$= 2 + 2 = 4$



# Rectilinear Kinematics: Accel. a function of velocity – a(v)

Given:

- A freighter moving at 8 knots when engines are stopped
- Deceleration $a = -kv^2$
- Speed reduces to 4 knots after ten minutes




Find:

(A) Speed of the ship as a function of time v(t)

(B) How far does the ship travel in the 10 minutes it takes to reduce the speed by 1/2 ?

Solution:

(A) With a, v & t parameters given/requested, use *a=dv/dt* form

$$a(v) = \frac{dv}{dt} \Rightarrow dt = \frac{dv}{a(v)} \Rightarrow \int_{t_i}^{t_f} dt = \int_{v_i}^{v_f} \frac{dv}{-kv^2}$$

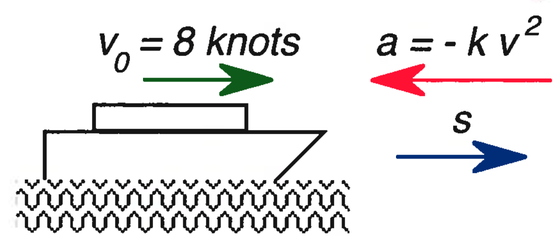
$$t_f = t(v) = \int_{v_i}^{v_f} \frac{dv}{a(v)} - t_i = \int_{8}^{v_f} \frac{dv}{-kv^2} + 0$$

$$\Rightarrow t_f = \left.\frac{1}{kv}\right|_8^{v_f} = \frac{1}{k}\left(\frac{1}{v_f} - \frac{1}{8}\right) \Rightarrow v_f = v(t_f) = \frac{8}{8kt_f + 1} \quad (knots)$$

# Rectilinear Kinematics: Accel. a function of velocity – a(v)

- Substituting BC's helps resolve the unknown constant *k*

$$t = \frac{10\ (min)}{60\ (min/hr)} = \frac{1}{6}\ hr\ ,\ v = 4\ \ knots$$




$$\Rightarrow v(1/6) = \frac{8}{8k(1/6)+1} = 4\ \ (knots)\quad \Rightarrow k = \frac{3}{4}\left(\frac{1}{nm}\right)$$

and the resulting expression for *speed* of the ship as a function of time v(t) is as follows

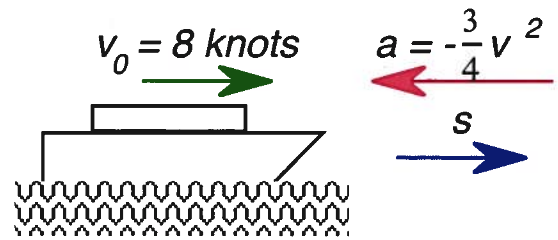
$$\underline{\underline{v_f = v(t) = \frac{8}{6t+1}}}\quad (knots)$$

- From here, there are two alternatives for resolving the second question

# Rectilinear Kinematics: Accel. a function of velocity – a(v)

(B) METHOD 1: Now, knowing the velocity as a function of time

$$v(t) = \frac{8}{6t+1} = \frac{ds}{dt}$$




the boat's position can be found by integration

$$\int_0^{s_f} ds = \int_0^{t_f} \frac{8}{6t+1}\, dt$$

$$s_f - 0 = \frac{4}{3}\ln(6t+1)\Big|_0^{t_f} = \frac{4}{3}\left(\ln(6t+1) - \ln(1)\right)$$

and the resulting expression for *position* of the ship as a function of time s(t)

$$s_f = s(t) = \frac{4}{3}\ln(6t+1)$$

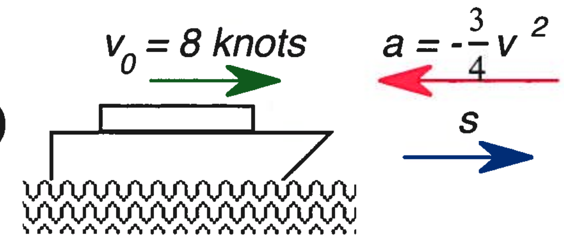
can now be used to find the particular displacement/distance at t=1/6 hr !

$$s(1/6) = \frac{4}{3}\ln(6(1/6)+1) = \underline{\underline{\frac{4}{3}\ln(2)}} \quad (nautical\ \ miles)$$

# Rectilinear Kinematics: Accel. a function of velocity – a(v)

(B) METHOD 2: With a, v & s parameters given/requested, use *ads=vdv* form

$$s_f = s(v) = \int_{v_i}^{v_f} \frac{vdv}{a(v)} + s_i \Rightarrow s(4) = \int_8^4 \frac{vdv}{\left(-3/4v^2\right)} + 0$$




and the boat's displacement (position?) can again be found by integration

$$s(4) = \frac{-4}{3}\int_8^4 \frac{dv}{v} = \frac{-4}{3} ln v\Big|_8^4 = \frac{-4}{3}\left(ln4 - ln8\right) = \frac{4}{3} ln\frac{8}{4}$$

and as was seen before

$$s(t = 1/6) \Rightarrow s(v = 4) = \underline{\underline{\frac{4}{3}\ \ln(2)}} \quad (nautical\ \ miles)$$

**Q.E.D.**

# 2D Curvilinear Kinematics Summary:

- **Position**

$$\underline{\mathbf{r}}(t) = x(t)\underline{\mathbf{i}} + y(t)\underline{\mathbf{j}}$$

$$= r(t)\underline{\mathbf{e}}_r$$

$$?\,path\,?$$

- **Velocity**

$$\underline{\mathbf{v}}(t) = \dot{\underline{\mathbf{r}}}(t) = \dot{x}\,\underline{\mathbf{i}} + \dot{y}\,\underline{\mathbf{j}}$$

$$= v\underline{\mathbf{e}}_t = \dot{s}\,\underline{\mathbf{e}}_t$$

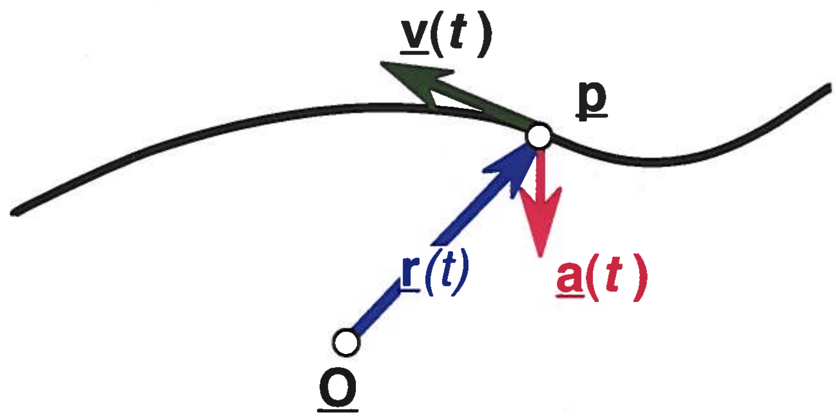
$$= \dot{r}\,\underline{\mathbf{e}}_r + r\dot{\theta}\,\underline{\mathbf{e}}_\theta$$

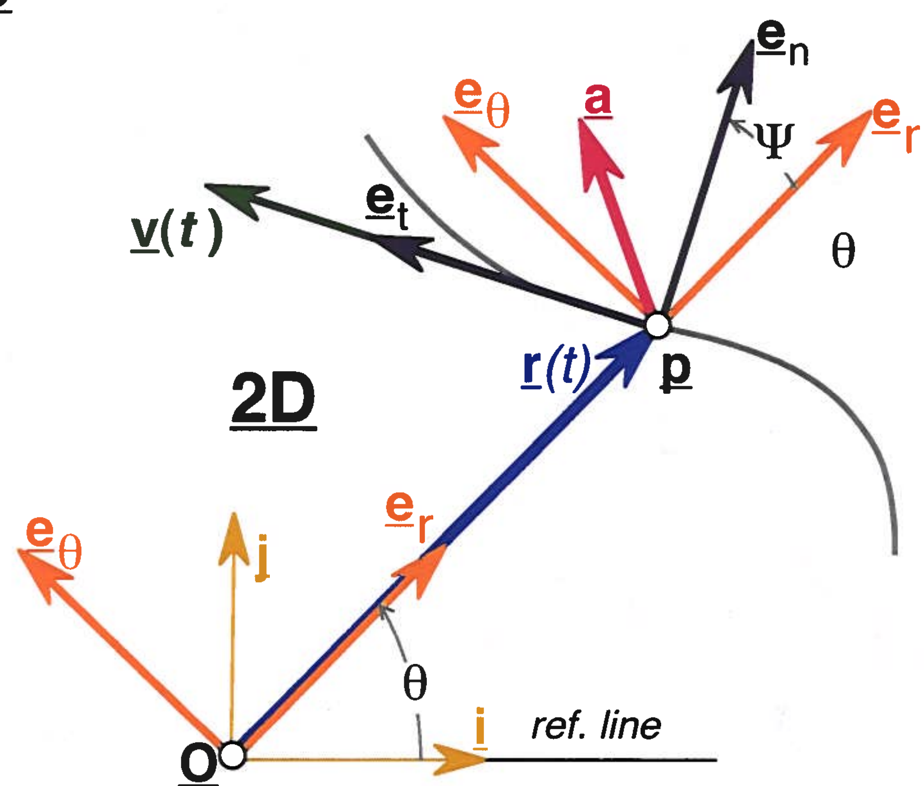
- **Acceleration**

$$\underline{\mathbf{a}}(t) = \dot{\underline{\mathbf{v}}}(t) = \ddot{\underline{\mathbf{r}}}(t) = \ddot{x}\,\underline{\mathbf{i}} + \ddot{y}\,\underline{\mathbf{j}}$$

$$= \ddot{s}\,\underline{\mathbf{e}}_t + \rho\dot{\theta}^2\,\underline{\mathbf{e}}_n = \dot{v}\,\underline{\mathbf{e}}_t + \frac{v^2}{\rho}\underline{\mathbf{e}}_n$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\underline{\mathbf{e}}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{\mathbf{e}}_\theta$$



# 2D Curvilinear Motion: Coordinates & Conversions

- Cartesian <-> Polar <-> Path

$$\underline{\mathbf{r}}(t) = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} = r\underline{\mathbf{e}}_r$$

$$\underline{\mathbf{e}}_r = \cos\theta\underline{\mathbf{i}} + \sin\theta\underline{\mathbf{j}} = \frac{x}{r}\underline{\mathbf{i}} + \frac{y}{r}\underline{\mathbf{j}}$$

$$\underline{\mathbf{e}}_\theta = \underline{\mathbf{k}} \times \underline{\mathbf{e}}_r = \cos\theta\underline{\mathbf{j}} - \sin\theta\underline{\mathbf{i}}$$

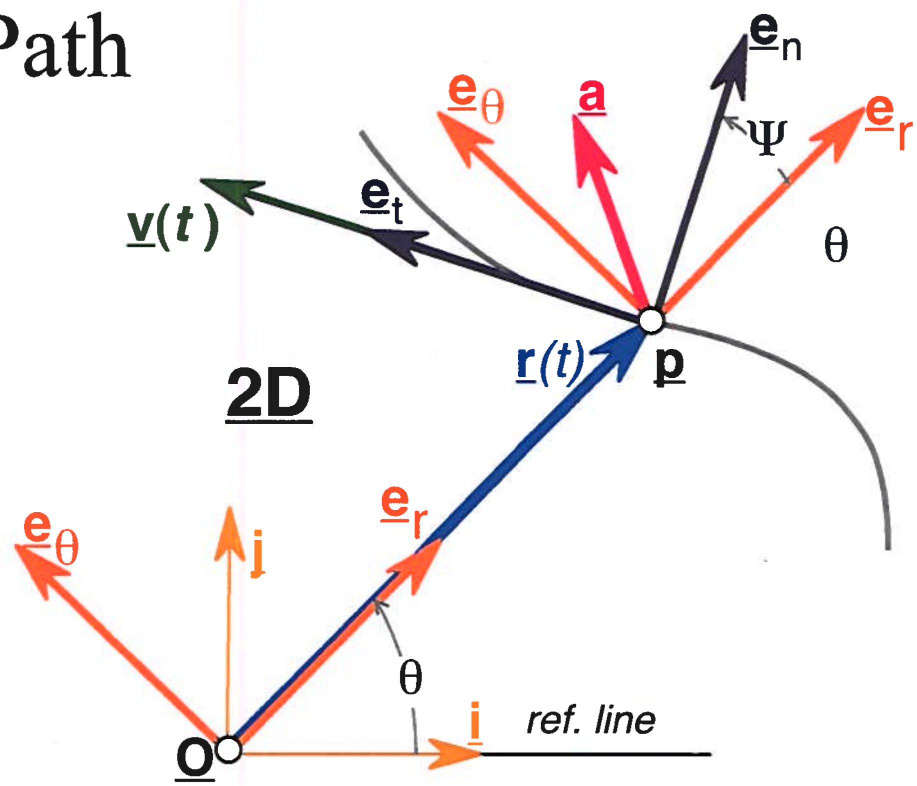
$$\underline{\mathbf{i}} = \cos\theta\underline{\mathbf{e}}_r - \sin\theta\underline{\mathbf{e}}_\theta$$

$$\underline{\mathbf{j}} = \underline{\mathbf{k}} \times \underline{\mathbf{i}} = \cos\theta\underline{\mathbf{e}}_\theta + \sin\theta\underline{\mathbf{e}}_r$$

$$\underline{\mathbf{v}}(t) = \dot{\underline{\mathbf{r}}}(t) = \dot{x}\underline{\mathbf{i}} + \dot{y}\underline{\mathbf{j}} = v\underline{\mathbf{e}}_t = \dot{s}\underline{\mathbf{e}}_t$$

$$\underline{\mathbf{e}}_t = \frac{\underline{\mathbf{v}}}{|\underline{\mathbf{v}}|} = \frac{\dot{x}}{v}\underline{\mathbf{i}} + \frac{\dot{y}}{v}\underline{\mathbf{j}}$$

$$\underline{\mathbf{e}}_n = \underline{\mathbf{k}} \times \underline{\mathbf{e}}_t = \frac{\dot{x}}{v}\underline{\mathbf{j}} - \frac{\dot{y}}{v}\underline{\mathbf{i}}$$




$$\underline{\mathbf{i}} = \cos\Psi\underline{\mathbf{e}}_t - \sin\Psi\underline{\mathbf{e}}_n$$

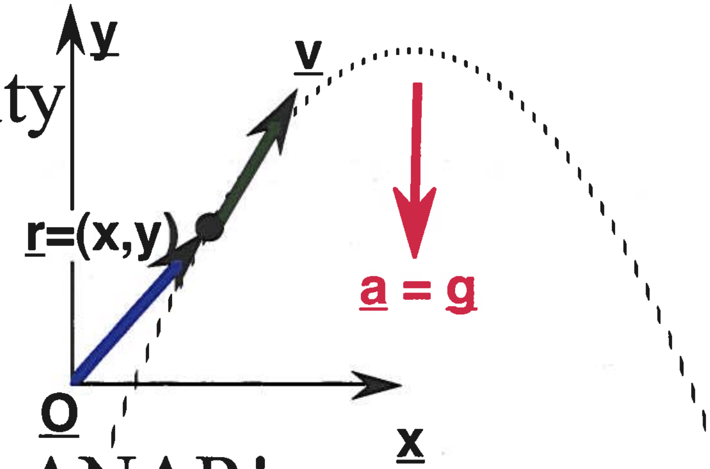
$$\underline{\mathbf{j}} = \underline{\mathbf{k}} \times \underline{\mathbf{i}} = \cos\Psi\underline{\mathbf{e}}_n + \sin\Psi\underline{\mathbf{e}}_t$$

## Curvilinear Motion: Cartesian Coordinates

- Projectile Motion
  - Scale w.r.t. earth such that gravity **g** is ~constant

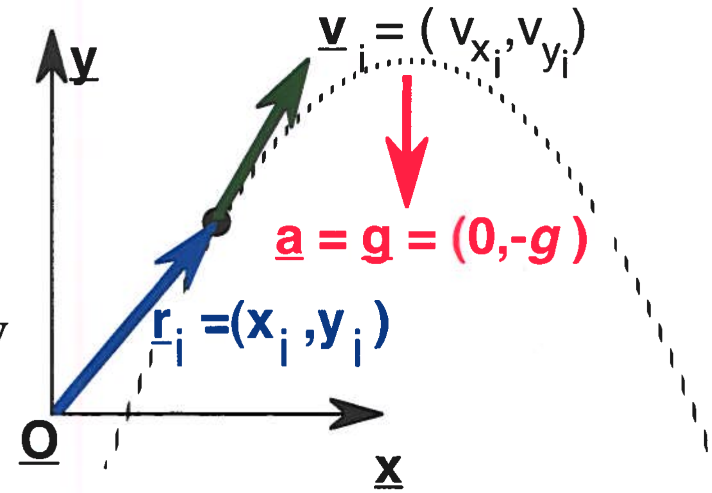
    $|g| = 32.2 \text{ ft/s}^2 = 9.81 \text{ m/s}^2$
  - Neglect any air resistance
  - Motion is PARABOLIC thus PLANAR!
  - Typically align
    - *y*-axis along gravity vector
    - *x*-axis horizontal in direction of motion

      $\underline{\mathbf{a}}(t) = 0\underline{\mathbf{i}} - g\,\underline{\mathbf{j}} = [0, -g]$
    - *z component drops out!*
  - Integrate rectilinear relations
    - Two (2) scalar relations
    - One VECTOR relationship



# Curvilinear Motion: Projectile Motion

- Typical P.M. queries
  - Max Height
  - Max Range
  - Time @ some place along trajectory
  - Later w/ Path & Polar Coord
    - Velocity (speed,direction/tangent)
    - Curvature, rate of speed change ….




$$\underline{\mathbf{a}}(t) = 0\underline{\mathbf{i}} - g\,\underline{\mathbf{j}} = [0, -g]$$

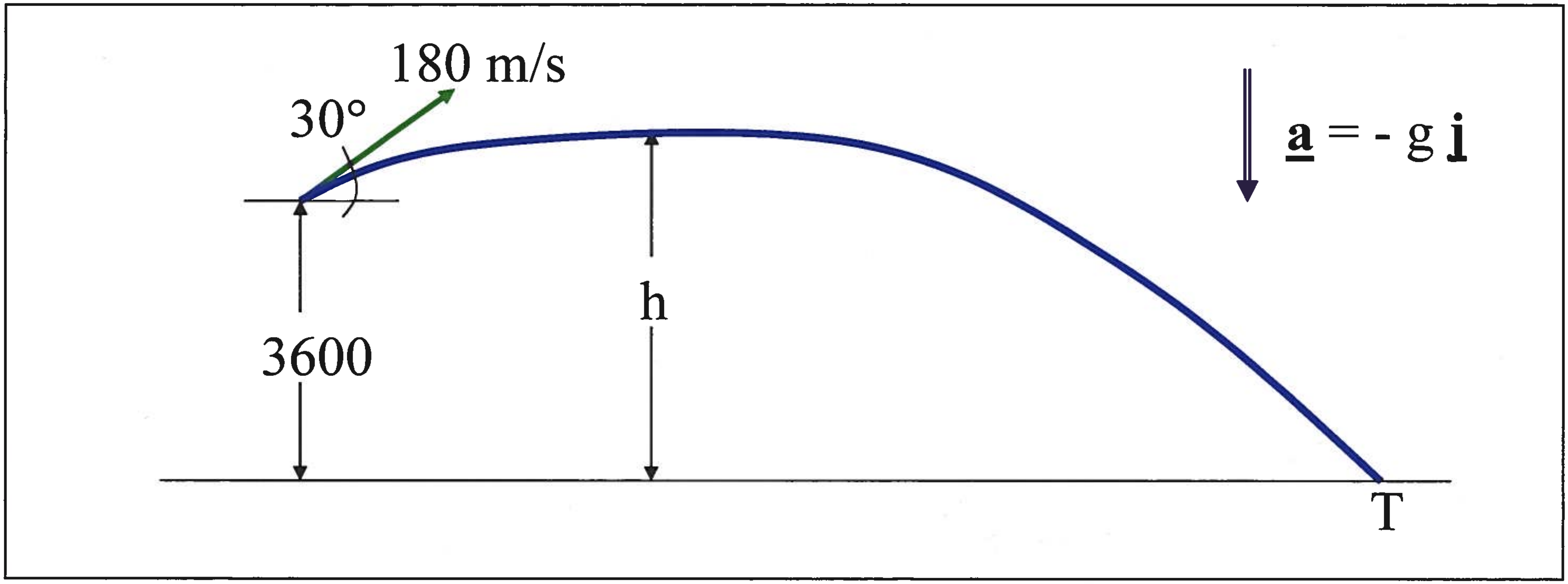
$$\Rightarrow \underline{\mathbf{v}}_f = \underline{\mathbf{a}}(t_f - t_i) + \underline{\mathbf{v}}_i \qquad \Rightarrow \underline{\mathbf{r}}_f = \frac{\underline{\mathbf{a}}}{2}(t_f - t_i)^2 + \underline{\mathbf{v}}_i(t_f - t_i) + \underline{\mathbf{r}}_i$$

$$v_{x_f} = 0(t_f - t_i) + v_{x_i} = v_{x_i} \qquad x_f = v_{xi}(t_f - t_i) + x_i$$

$$v_{y_f} = \text{-}g(t_f - t_i) + v_{y_i} \qquad y_f = \frac{-g}{2}(t_f - t_i)^2 + v_{y_i}(t_f - t_i) + y_i$$

- Reconsider problems w/ different axes placement/orientation

**Given: launch at 3600 m altitude $v_o$ = 180 m/s angle 30°**




$\ddot{x} = 0$ $\qquad$ $\ddot{y} = -9.81$

$\dot{x} = 180 (\cos 30) = 156$ $\qquad$ $\dot{y} = 180 (\sin 30) - 9.81\, t = 90 - 9.81\, t$

$x = 156\, t$ $\qquad$ $y = 90\, t - 4.905\, t^2 + 3600$

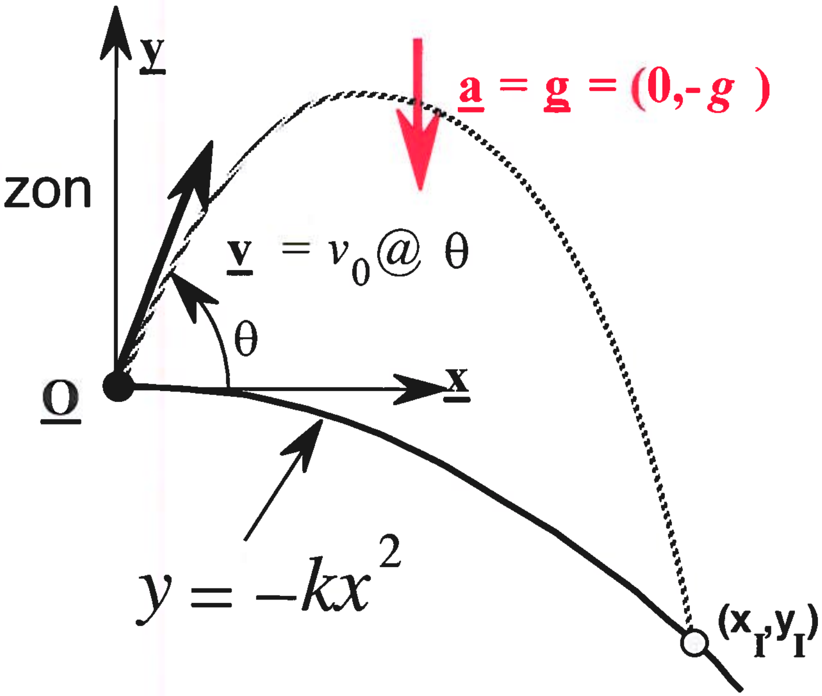
for h set $\dot{y} = 0 \Longrightarrow t = 9.17 \Longrightarrow h = y = 4013$ m

for $t_T$ set $y = 0 \Longrightarrow t = 37.8$

# Curvilinear Kinematics: Projectile Motion example ref: Hibbler 12-104

Given:

- Figure shown w/ ground $y = -kx^2$
- $t_0=0$, $(x_0,y_0)=\underline{\mathbf{0}}$, $\underline{\mathbf{v}}_0=v_0$ @ θ above horizon




Find: In terms of $v_0$, $\theta$ & $k$

- (A) The location at impact $(x_I,y_I)$
- (B) Velocity & Speed @ impact, $\underline{\mathbf{v}}_I$, $v_I$
- (C) Elapsed time @ impact, $t_I$

Solution:

- 2D projectile motion
- Get expressions for $v_x(t)$,$v_y(t)$ then $x(t)$, $y(t)$

$$\Rightarrow \underline{\mathbf{v}}_f(t) = \underline{\mathbf{a}}_c\left(t_f - t_i\right) + \underline{\mathbf{v}}_i \qquad \Rightarrow \underline{\mathbf{r}}_f = \frac{\underline{\mathbf{a}}_c}{2}\left(t_f - t_i\right)^2 + \underline{\mathbf{v}}_i\left(t_f - t_i\right) + \underline{\mathbf{r}}_i$$

- Substitute into ground constraint expression
  - Solve for time of impact $(t_I)$
- With $t_I$ known, substitute & solve for $(x_I,y_I)$

# Curvilinear Kinematics: Projectile Motion

- IC's => $t_0=0$, $(x_0,y_0)=\underline{0}$, $\underline{\mathbf{v}}_0=v_0$ @ $\theta$

$$\underline{\mathbf{a}}(t) = 0\underline{\mathbf{i}} - g\,\underline{\mathbf{j}} = [0,-g]$$

$$\Rightarrow (B)\quad \underline{\underline{\mathbf{v}}_I = \underline{\mathbf{a}}(t_I - 0) + \underline{\mathbf{v}}_0} = \left[v_{x_I}, v_{y_I}\right]$$

$$v_{x_I} = v_0 \cos\theta$$

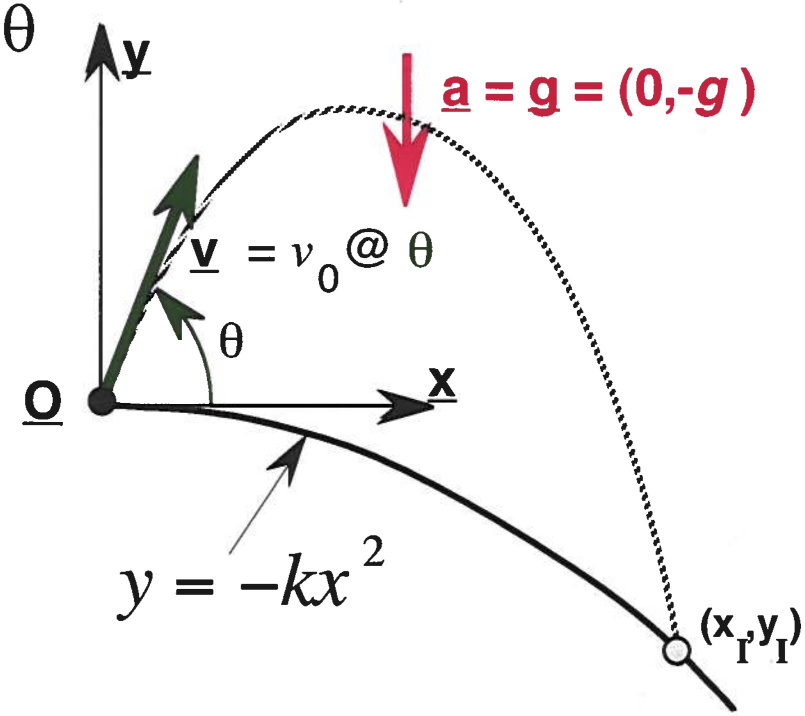
$$v_{y_I} = -gt_I + v_0 \sin\theta$$

- Speed

$$\dot{s} = v = \sqrt{v_{x_I}^2 + v_{y_I}^2}$$

$$= \sqrt{(v_0\cos\theta)^2 + (-gt_I + v_0\sin\theta)^2}$$

$$= \sqrt{v_0^2 - 2gv_0\sin\theta t_I + (gt_I)^2}$$



# Curvilinear Kinematics: Projectile Motion

$$\Rightarrow (B) \quad \underline{\underline{\mathbf{v}}_I = \underline{\mathbf{a}}(t_I - 0) + \underline{\mathbf{v}}_0} = \left[ v_{x_I}, v_{y_I} \right]$$

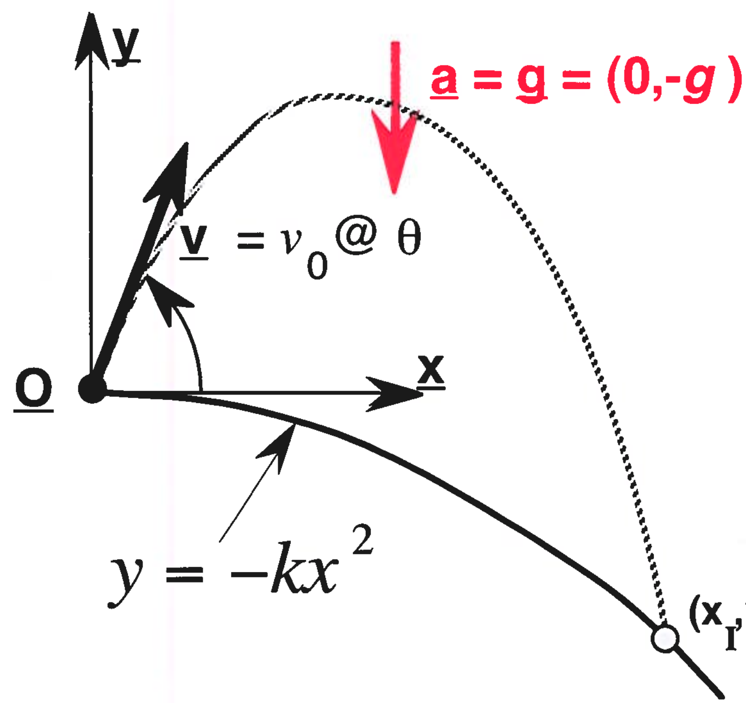
$$\Rightarrow (A) \quad \underline{\underline{\mathbf{r}}_I = \frac{\underline{\mathbf{a}}}{2}(t_I - 0)^2 + \underline{\mathbf{v}}_0(t_I - 0) + \underline{\mathbf{0}}}$$

$$\begin{cases} x_I = v_0 \cos\theta \; t_I \\ y_I = \dfrac{-g}{2} t_I^{\,2} + v_0 \sin\theta \; t_I \end{cases}$$

$$y = -kx^2$$

$$\frac{-g}{2} t_I^{\,2} + v_0 \sin\theta \; t_I = -k\left(v_0 \cos\theta \; t_I\right)^2$$

$$\Rightarrow (C) \quad t_I = \frac{2 v_0 \sin\theta}{g - 2k\left(v_0 \cos\theta\right)^2}, \quad t_I = 0$$

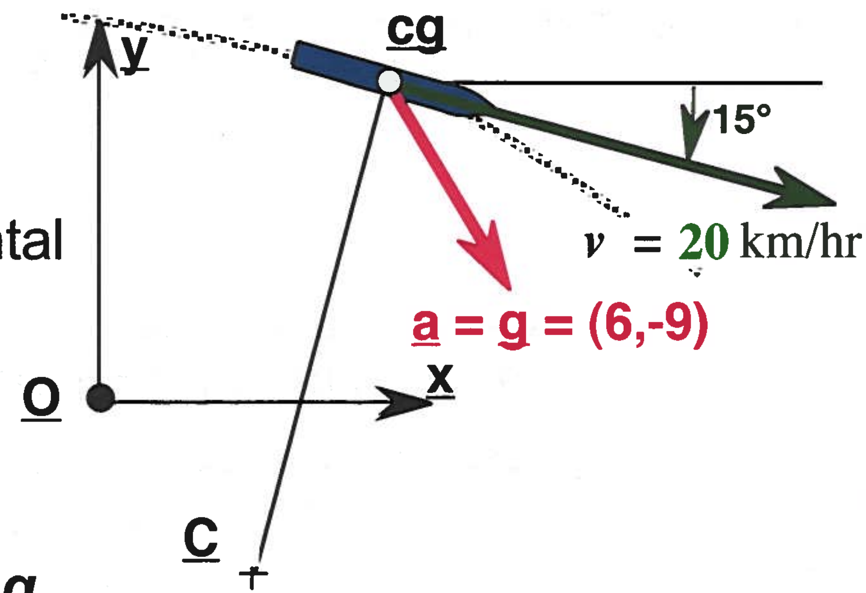



– Substitute value for $t_I$ into position, velocity & speed relations for solution

# Path Coord. Example ref: Meriam&Kraige 2-8

Given:

- A rocket at high altitude with
- $\underline{a}_0$=6$\underline{i}$-9$\underline{j}$ (m/s$^2$)
- $\underline{v}_0$=20 (km/hr) @ 15 below horizontal




Find: At instant given

(A) The ***normal*** & ***tangential*** *accelerations*

(B) Rate at which ***speed*** is ***increasing***

(C) ***Radius of curvature*** of the path

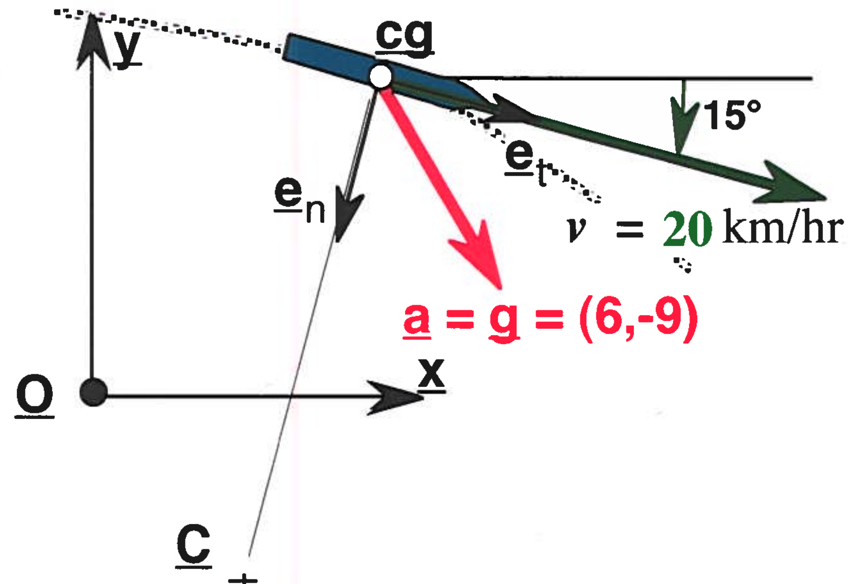
(D) Angular ***rotation rate*** of the radial from CG to center of curvature

Solution:

- "High altitude" means negligible air resistance
- Interested only at this instant (NO Integration required)
- ***Cartesian*** specified, asking for ***Path*** coord parameters
- **V** given is TANGENT TO THE PATH
  - Use this to relate *path* to *cartesian* coordinates

# Path Coord. Example ref: Meriam&Kraige 2-8

Solution (cont'd):




$$\underline{e}_t = \frac{\underline{v}}{|\underline{v}|} = \frac{\underline{v}}{v} = \cos 15°\underline{i} - \sin 15°\underline{j}$$

$$\underline{e}_n = \pm\left(\underline{k} \times \underline{e}_t\right) \qquad \textit{(2D shortcut!)}$$

$$= -\cos 15°\underline{j} - \sin 15°\underline{i}$$

(A) $a_n$ & $a_t$=?

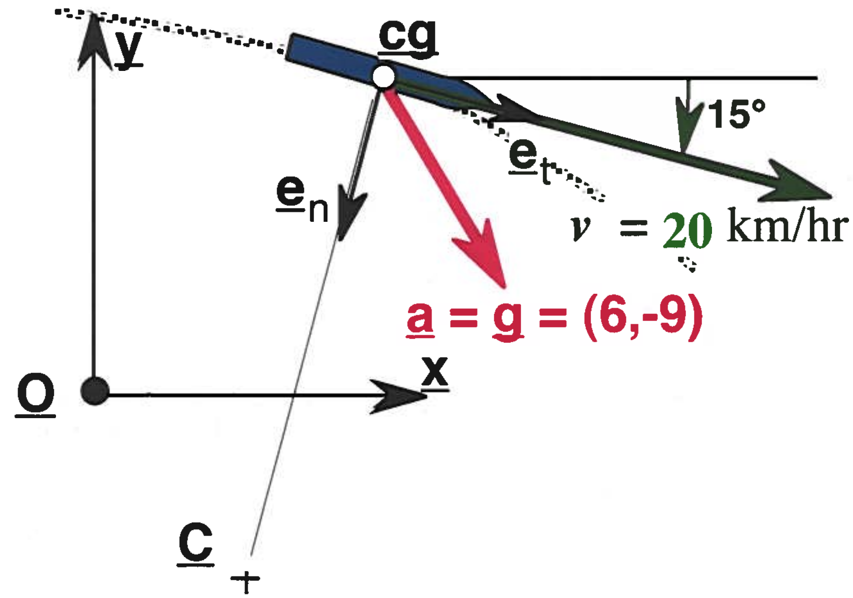
$$|\underline{a}_t| = \underline{a} \bullet \underline{e}_t = \left(6\underline{i} - 9\underline{j}\right) \bullet \left(\cos 15°\underline{i} - \sin 15°\underline{j}\right) = \underline{\underline{8.12 \ (m/s^2) = a_t}}$$

$$|\underline{a}_n| = \underline{a} \bullet \underline{e}_n = \left(6\underline{i} - 9\underline{j}\right) \bullet \left(-\cos 15°\underline{j} - \sin 15°\underline{i}\right) = \underline{\underline{7.14 \ (m/s^2) = a_n}}$$

(B) $\dot{v} = ??$ $\quad \dot{v} = |\underline{a}_t| = \underline{\underline{8.12 \ (m/s^2)}}$

(C) ρ=? $a_n = \frac{v^2}{\rho} \Rightarrow \rho = \frac{v^2}{a_n} = \frac{(20 \quad km/hr)^2}{7.14(m/s^2)}\left(\frac{1hr}{3600s} * \frac{10^3 m}{km}\right)^2 = \underline{\underline{4.32(10^6)m}}$

# Path Coord. Example ref: Meriam&Kraige 2-8




Solution (cont'd):

(D) $\dot{\theta} = ??$

- Look either at $a_n$ or velocity

$$a_n = \rho\dot{\theta}^2$$

$$\Rightarrow \dot{\theta} = \sqrt{\frac{a_n}{\rho}} = \sqrt{\frac{7.14\ (m/s^2)}{4.32*10^6(m)}} = \underline{\underline{12.9*10^{-4}\frac{1}{s}}}$$

$$v = \rho\dot{\theta}$$

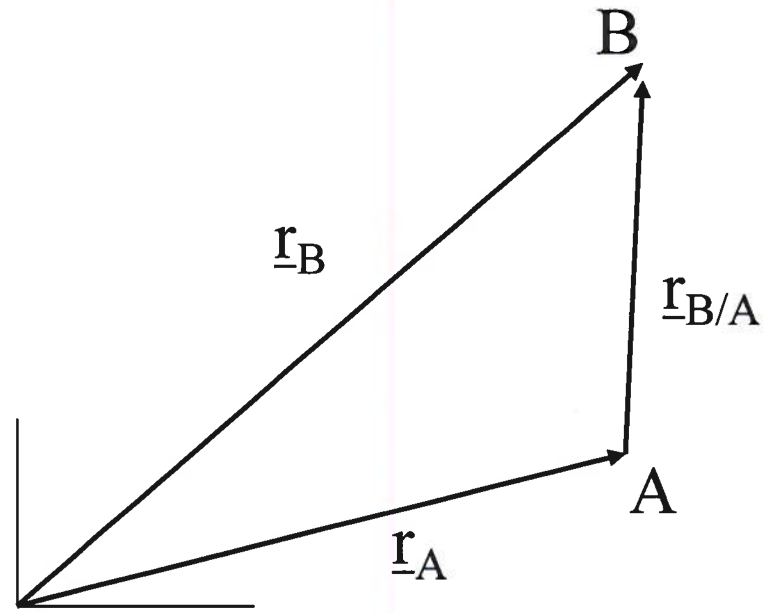
$$\Rightarrow \dot{\theta} = \frac{v}{\rho} = \frac{20\quad km/hr}{4.32*10^6(m)}\left(\frac{1hr}{3600s}*\frac{10^3 m}{km}\right) = \underline{\underline{12.9*10^{-4}s^{-1}}}$$

# Relative Motion

$\underline{\mathbf{r}}_B = \underline{\mathbf{r}}_A + \underline{\mathbf{r}}_{B/A}$

$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\mathbf{v}}_{B/A}$

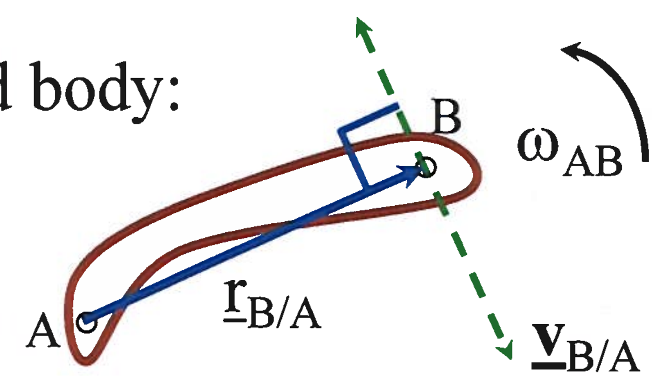
$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\mathbf{a}}_{B/A}$




## Special Case: Rigid Bodies

When A & B are two points on the same rigid body:

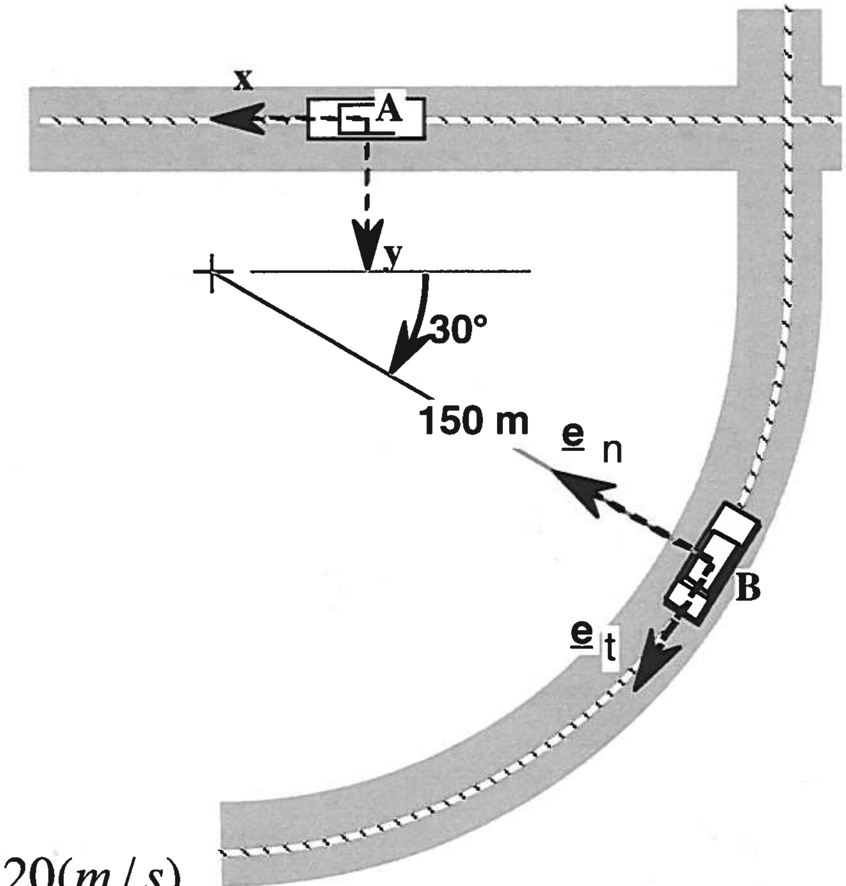
- the relative motion is circular
- $\underline{\mathbf{v}}_{B/A}$ is perpendicular (⊥) to $\underline{\mathbf{r}}_{B/A}$ & $|\underline{\mathbf{v}}_{B/A}| = |\omega_{AB}\, AB|$




$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \omega_{AB}\ \underline{\mathbf{k}} \times AB\underline{\mathbf{u}}_{B/A}$$

$$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \alpha_{AB}\,\underline{\mathbf{k}} \times AB\underline{\mathbf{u}}_{B/A} - (\omega_{AB})^2\, AB\underline{\mathbf{u}}_{B/A}$$

# Relative Motion: ref ~Meriam & Kraige 2/13




## Given:

- Two cars **A** & **B** at the instant shown

  $\underline{\mathbf{v}}_A = 72\ \underline{\mathbf{i}}$ km/hr $\quad\quad \underline{\mathbf{a}}_A = 1.2\ \underline{\mathbf{i}}$ m/s$^2$

  $\underline{\mathbf{v}}_B = 54\ \underline{\mathbf{e}}_t$ km/hr, *constant speed*

## Find:

(A) $\underline{\mathbf{v}}_{B/A} = ?$ (B) $\underline{\mathbf{a}}_{B/A} = ?$

## Solution:

- Convert to consistent units

$$(km/hr)*\frac{1}{3.6} = (m/s) \quad \Rightarrow \quad \begin{aligned} v_A &= 72(km/hr) = 20(m/s) \\ v_B &= 54(km/hr) = 15(m/s) \end{aligned}$$

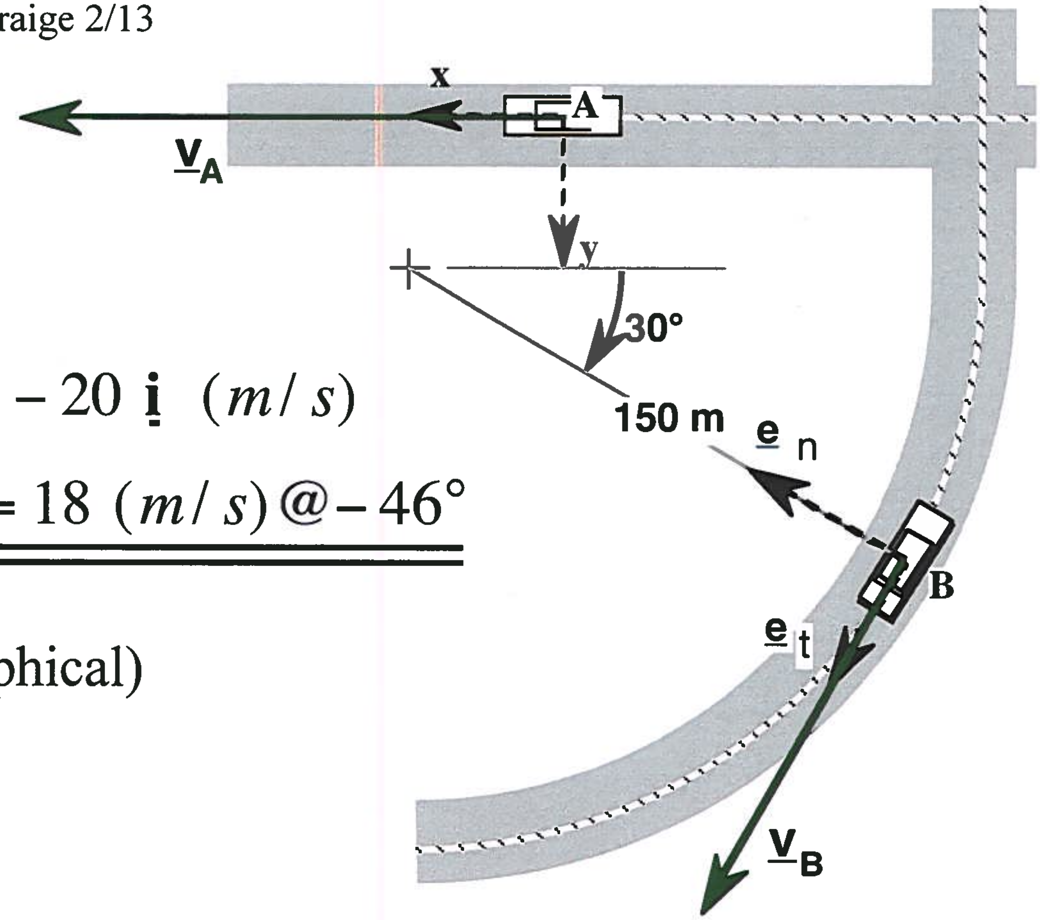
- Motion RELATIVE TO **A** of interest
- Two coordinate axes are used
  - Simplifies $\underline{\mathbf{v}}$ & $\underline{\mathbf{a}}$ definitions
  - Illustrates “coordinate conversion” for expressing answers “in terms of” a unified set.

$$\underline{\mathbf{e}}_n = \cos 30°\underline{\mathbf{i}} - \sin 30°\underline{\mathbf{j}}$$

$$\underline{\mathbf{e}}_t = \underline{\mathbf{k}} \times \underline{\mathbf{e}}_n = \cos 30°\underline{\mathbf{j}} + \sin 30°\underline{\mathbf{i}}$$

# Relative Motion: ref ~Meriam & Kraige 2/13

## (A) Relative Velocity




$$\underline{\mathbf{v}}_{B/A} = \underline{\mathbf{v}}_B - \underline{\mathbf{v}}_A$$

$$= 15\underline{\mathbf{e}}_t - 20\ \underline{\mathbf{i}}\ \ (m/s)$$

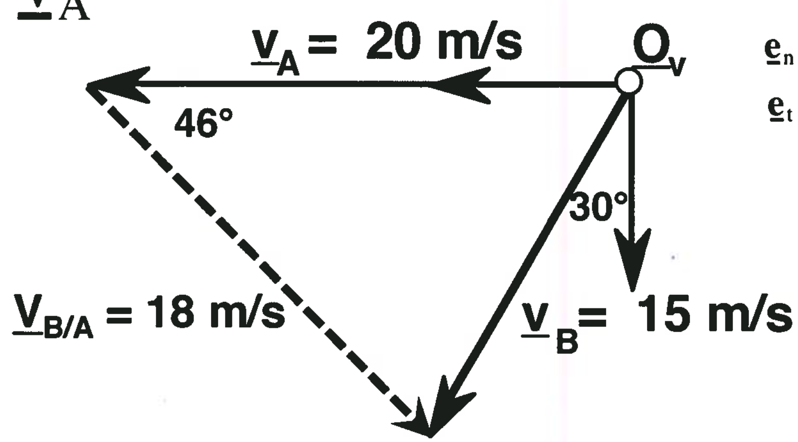
$$= 15(\sin 30°\underline{\mathbf{i}} + \cos 30°\underline{\mathbf{j}}) - 20\ \underline{\mathbf{i}}\ \ (m/s)$$

$$\underline{\mathbf{v}}_{B/A} = -12.5\underline{\mathbf{i}} + 13.0\underline{\mathbf{j}}\ (m/s) = 18\ (m/s) @ -46°$$

- Velocity Polygon Approach (Graphical)

$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\mathbf{v}}_{B/A}$$

$$\Rightarrow \underline{\mathbf{v}}_{B/A} = \underline{\mathbf{v}}_B - \underline{\mathbf{v}}_A$$




$$\underline{\mathbf{e}}_n = \cos 30°\underline{\mathbf{i}} - \sin 30°\underline{\mathbf{j}}$$

$$\underline{\mathbf{e}}_t = \underline{\mathbf{k}} \times \underline{\mathbf{e}}_n = \cos 30°\underline{\mathbf{j}} + \sin 30°\underline{\mathbf{i}}$$

# Relative Motion: ref ~Meriam & Kraige 2/13

(B) Relative Acceleration

$$\underline{\mathbf{a}}_A = 1.2\,\underline{\mathbf{i}} \quad (m/s^2)$$

$$\underline{\mathbf{a}}_B = \dot{v}\,\underline{\mathbf{e}}_t + \frac{v^2}{\rho}\,\underline{\mathbf{e}}_n \quad (m/s^2) = \frac{(15\ m/s)^2}{150\ m}\underline{\mathbf{e}}_n$$

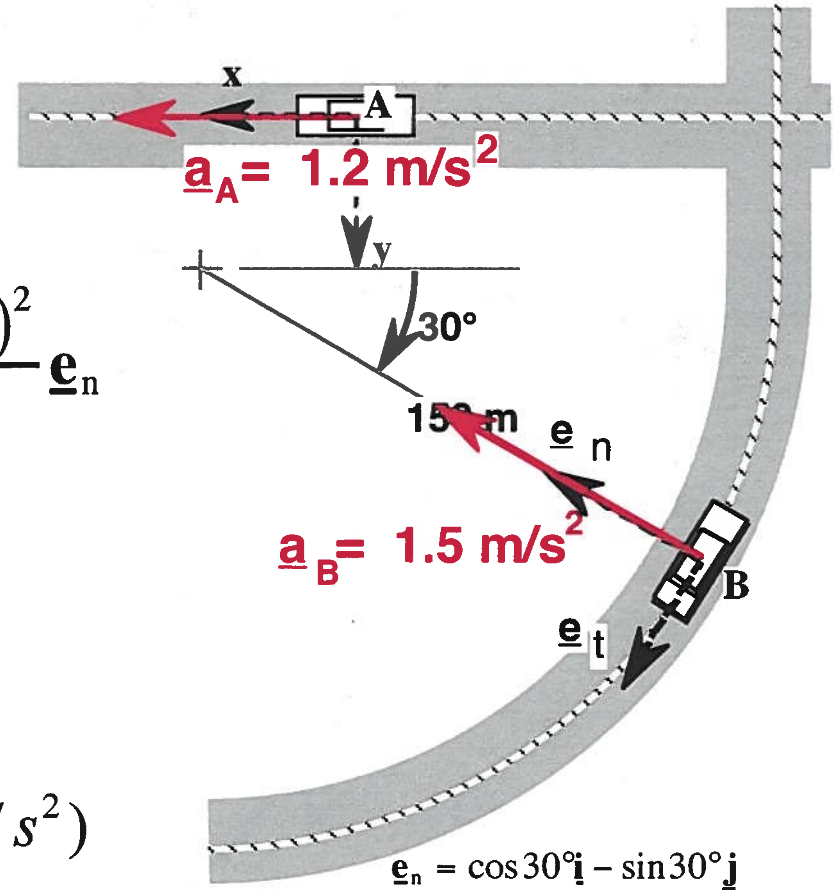
$$= 1.5\underline{\mathbf{e}}_n \quad (m/s^2)$$

$$\underline{\mathbf{a}}_{B/A} = \underline{\mathbf{a}}_B - \underline{\mathbf{a}}_A$$

$$= 1.5\,\underline{\mathbf{e}}_n - 1.2\underline{\mathbf{i}} \quad (m/s^2)$$

$$= 1.5(\cos 30^\circ \underline{\mathbf{i}} - \sin 30^\circ \underline{\mathbf{j}}) - 1.2\underline{\mathbf{i}} \quad (m/s^2)$$

$$\underline{\mathbf{a}}_{B/A} = 0.1\,\underline{\mathbf{i}} - 0.75\,\underline{\mathbf{j}}\,(m/s^2) = 0.76(m/s^2)@-82^\circ$$



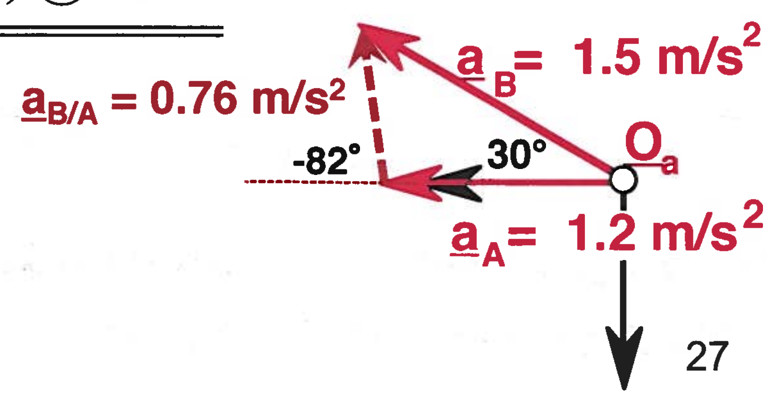

$$\underline{\mathbf{e}}_n = \cos 30^\circ \underline{\mathbf{i}} - \sin 30^\circ \underline{\mathbf{j}}$$

$$\underline{\mathbf{e}}_t = \underline{\mathbf{k}} \times \underline{\mathbf{e}}_n = \cos 30^\circ \underline{\mathbf{j}} + \sin 30^\circ \underline{\mathbf{i}}$$

- Acceleration Polygon (Graphical)

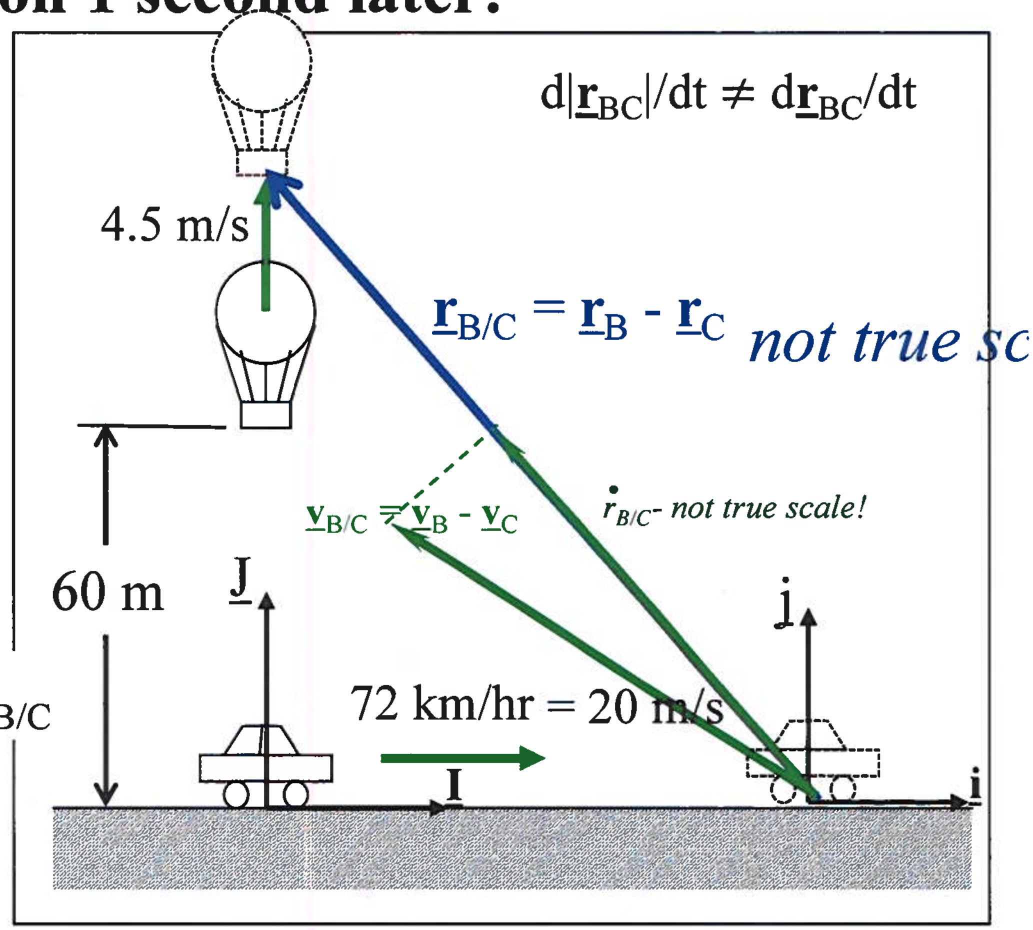
$$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\mathbf{a}}_{B/A}$$

$$\Rightarrow \underline{\mathbf{a}}_{B/A} = \underline{\mathbf{a}}_B - \underline{\mathbf{a}}_A$$



**Given:** A balloon at an altitude of 60 m is rising at steady rate of 4.5 m/s. A car passes below at constant speed of 72 kph.

**Find:** Relative rate of separation 1 second later:




$\underline{a}_C = 0 \qquad \underline{a}_B = 0 \quad (m/s^2)$

$\underline{v}_C = 20\underline{i} \qquad \underline{v}_B = 4.5\underline{j} \quad (m/s)$

$\underline{r}_C = 20t\,\underline{i} \qquad \underline{r}_B = (60 + 4.5t)\underline{j}\ (m)$

$$r_{B/C} = |\underline{r}_{B/C}| = \sqrt{\underline{r}_{B/C} \bullet \underline{r}_{B/C}}$$

$$r^2_{B/C} = (-20t)^2 + (60 + 4.5t)^2$$

$$2r_{B/C}\dot{r}_{B/C} = 2(-20t)(-20) + 2(60 + 4.5t)4.5$$

Evaluate @ t = 1 & divide through by 2 $r_{B/C}$

$$\dot{r}_{B/C} = 690.25/67.52 = 10.22(m/s)$$

Alternative Method (Vectors!)

=> Find the radial component of $\underline{V}_{B/C} = \dot{\underline{r}}_{B/C} = \dot{r}_{B/C}\underline{e}_{r_{B/C}} + r_{B/C}\dot{\theta}\underline{e}_{\theta_{B/C}}$

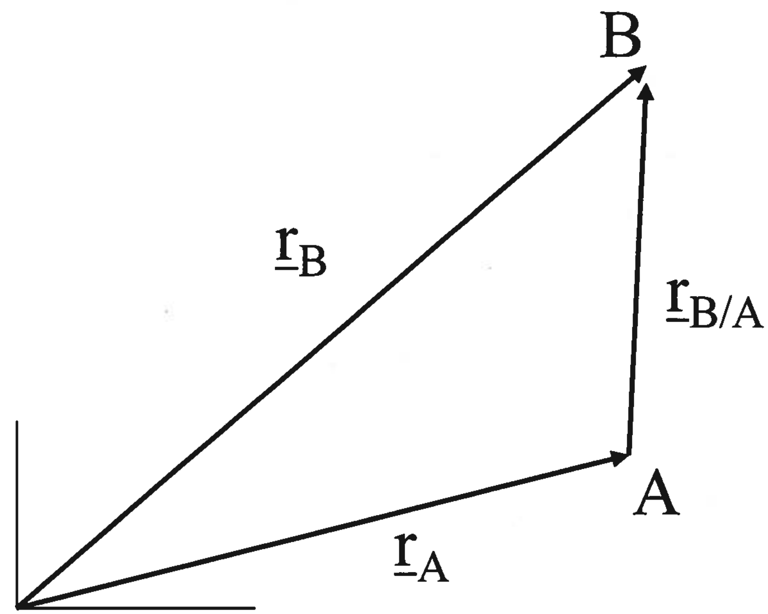
$$\dot{r}_{B/C} = \underline{V}_{B/C} \bullet \frac{\underline{r}_{B/C}}{|\underline{r}_{B/C}|} = (\underline{V}_B - \underline{V}_C) \bullet \frac{(\underline{r}_B - \underline{r}_C)}{|\underline{r}_{B/C}|} = (-20, 4.5) \bullet \frac{(-20, 64.5)}{|67.5|} = \frac{690.3}{67.5} = 10.2\,(m/s)$$

# Relative Motion

$$\underline{\mathbf{r}}_B = \underline{\mathbf{r}}_A + \underline{\mathbf{r}}_{B/A}$$

$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\mathbf{v}}_{B/A}$$

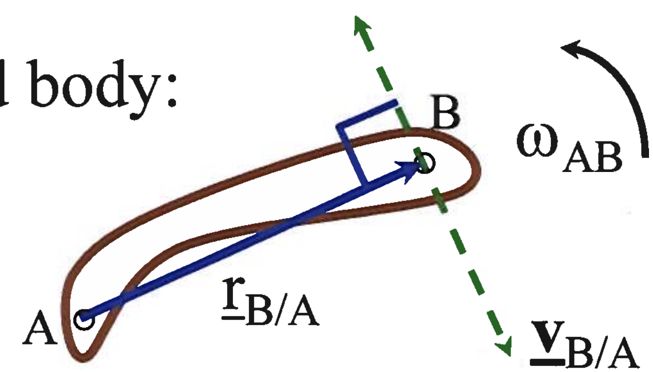
$$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\mathbf{a}}_{B/A}$$




## Special Case: Rigid Bodies

When A & B are two points on the same rigid body:

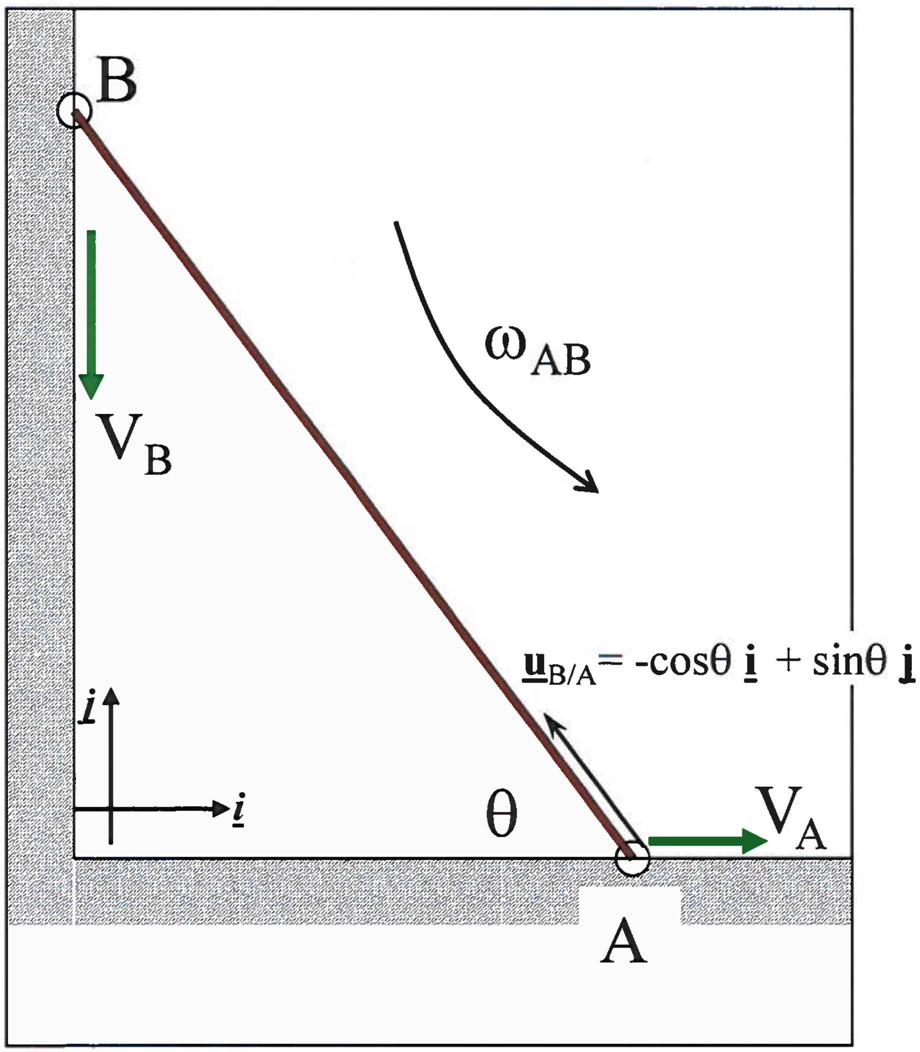
- the relative motion is circular
- $\underline{\mathbf{v}}_{B/A}$ is perpendicular (⊥) to $\underline{\mathbf{r}}_{B/A}$
  & $|\underline{\mathbf{v}}_{B/A}| = |\omega_{AB}\, AB|$




$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \omega_{AB}\, \underline{\mathbf{k}} \times AB\underline{\mathbf{u}}_{B/A}$$

$$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \alpha_{AB}\, \underline{\mathbf{k}} \times AB\underline{\mathbf{u}}_{B/A} - (\omega_{AB})^2\, AB\underline{\mathbf{u}}_{B/A}$$

## Two points on a rigid body:




$$\underline{v}_B = \underline{v}_A + \underline{v}_{B/A}$$

$$v_B\ \underline{j} = v_A\ \underline{i} + \omega_{AB}\ \underline{k} \times AB\ \underline{u}_{B/A}$$

$$v_B\ \underline{j} = v_A\ \underline{i}$$

$$- AB\ \omega_{AB}(\sin\theta\ \underline{i} + \cos\theta\ \underline{j})$$

Equating $\underline{i}$ & $\underline{j}$ components:

$$\underline{i} \rightarrow v_A - AB\ \omega_{AB} \sin\theta = 0$$

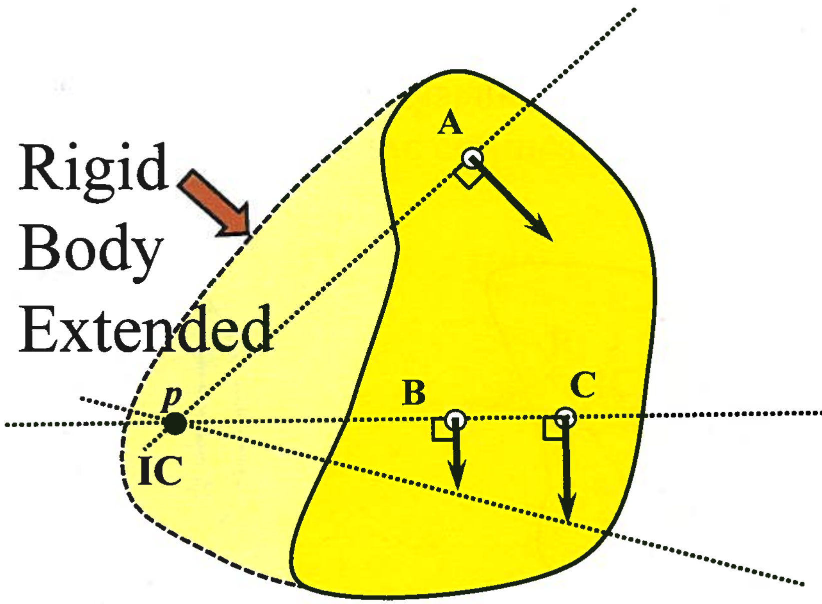
$$\underline{j} \rightarrow v_B = AB\ \omega_{AB} \cos\theta$$

$$\frac{v_A}{v_B} = \frac{AB\ \omega_{AB} \sin\theta}{AB\ \omega_{AB} \cos\theta}$$

$$\frac{v_A}{v_B} = \frac{\sin\theta}{\cos\theta}$$

# Instant Centers (velocity)

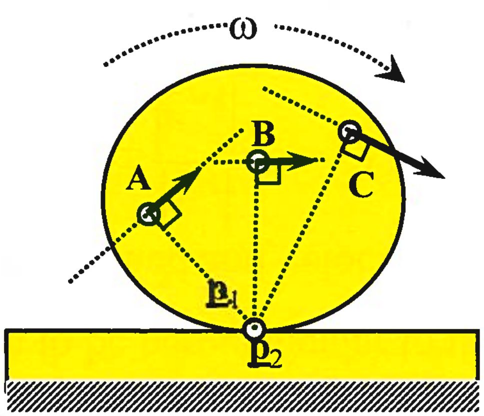
On every rigid body in general plane motion there exists a point **P** where $\underline{V}_P=0$! It is known as **instantaneous center (IC) of zero velocity** or **instantaneous center of rotation (ICR)**




### How to Locate IC?

1. Every point's velocity vector is perpendicular to ⊥ its relative position vector from the instant center
2. Its speed (velocity magnitude) is proportional to its distance from IC

$$\omega = \pm \frac{|\underline{\mathbf{v}}_A|}{r_{IC-A}} = \frac{|\underline{\mathbf{v}}_B|}{r_{IC-B}} = \ldots.$$




**Rolling** disk/tire (**no slip**!!)

At any instant, @ **point of contact** $\Rightarrow \underline{\mathbf{v}}_{p_1/p_2} = 0!$

If $\mathbf{p}_2$ on the ground $\Rightarrow \underline{\mathbf{v}}_{p_2} = 0 \Rightarrow \underline{\mathbf{v}}_{p_1} = 0$

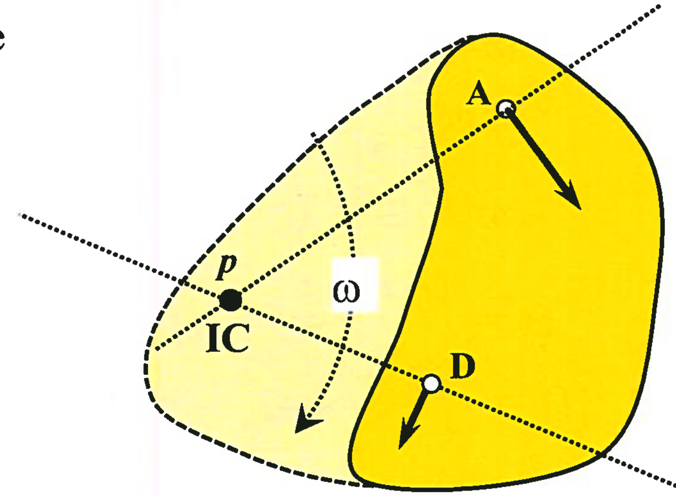
$$\underline{\mathbf{v}}_A = \underline{\mathbf{v}}_{p1} + \underline{\omega}\times\underline{r}_{pA} = \underline{0} + \underline{\omega}\times\underline{r}_{pA} \qquad |\underline{\mathbf{v}}_A| = r_{pA}\omega$$

$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_{p1} + \underline{\omega}\times\underline{r}_{pB} = \underline{0} + \underline{\omega}\times\underline{r}_{pB} \qquad |\underline{\mathbf{v}}_B| = r_{pB}\omega$$

$$\underline{\mathbf{v}}_C = \underline{\mathbf{v}}_{p1} + \underline{\omega}\times\underline{r}_{pC} = \underline{0} + \underline{\omega}\times\underline{r}_{pC} \qquad |\underline{\mathbf{v}}_C| = r_{pC}\omega$$

**Knowing location of IC => Very useful tool!**

The direction of velocity for all points on the rigid body are known to be perpendicular to the line from **IC** to that point



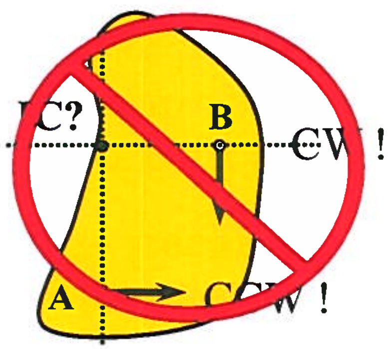

- If **IC** located and velocity of any one point is known:

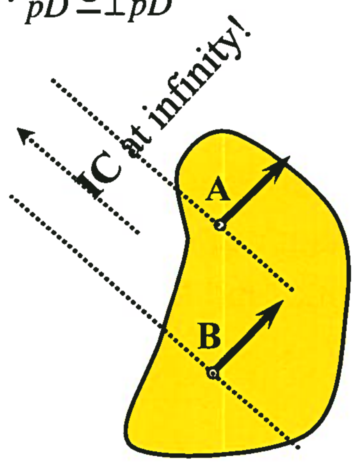
$$\omega = \pm \frac{|\underline{\mathbf{v}}_A|}{|\underline{r}_{pA}|} \qquad \text{CW } \sout{\text{or CCW}}\ ?$$

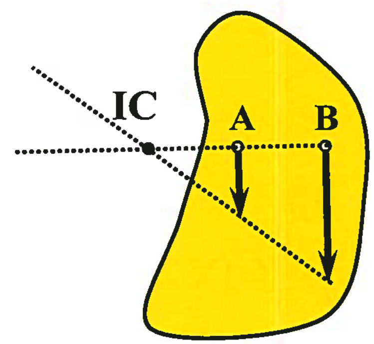
- If **IC** located and magnitude of $\omega$ is known, the velocity of any point D is:

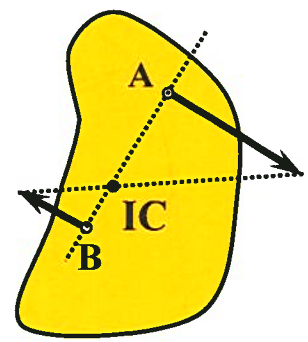
$$\underline{\mathbf{v}}_D = \underline{\omega} \times \underline{r}_{pD} = \omega r_{pD} \underline{\mathbf{e}}_{\perp pD}$$

Special cases:



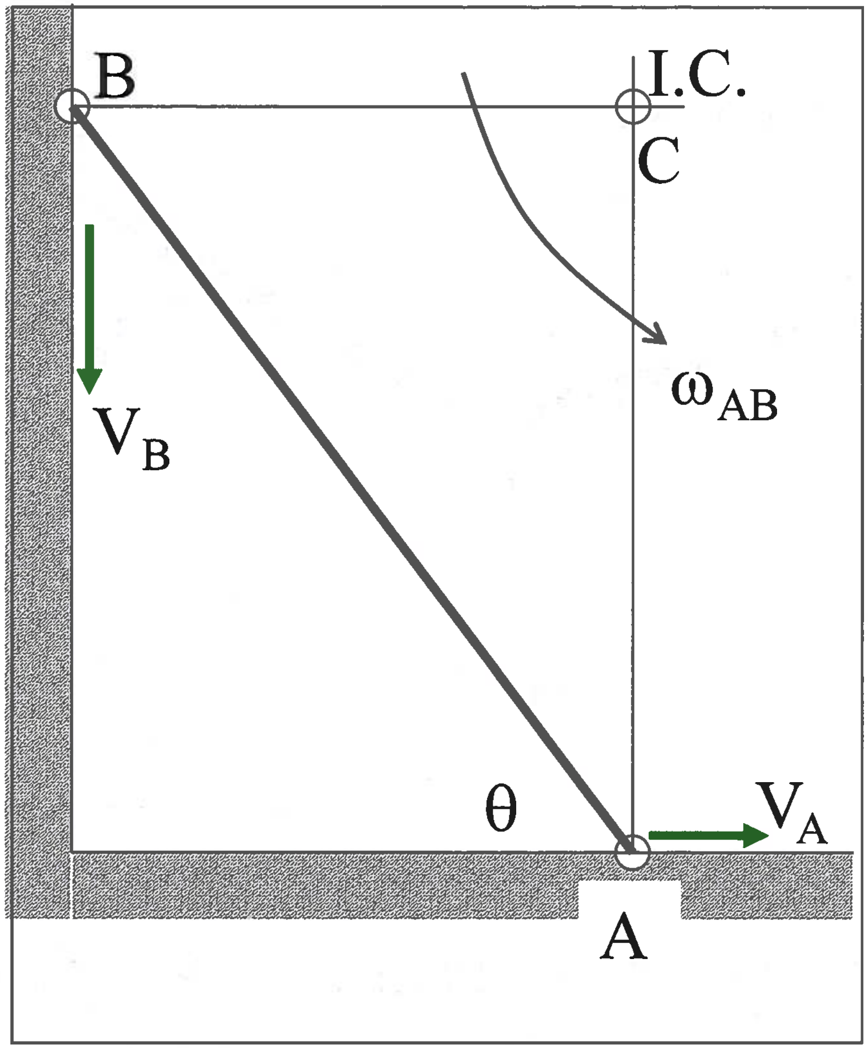

Construction lines are parallel, not collinear

Mathematically the IC is at infinity!
*Pure Translation!*

$$\underline{\mathbf{v}}_A = \underline{\mathbf{v}}_B \Rightarrow \omega = \pm \frac{|\underline{\mathbf{v}}_A|}{\infty} = 0$$

The construction lines are collinear!
Speed is proportional to distance from IC.

$$\omega = \pm \frac{|\underline{\mathbf{v}}_A|}{r_{pA}} = \frac{|\underline{\mathbf{v}}_B|}{r_{pB}}$$




# Using Instant Centers (IC):

$$V_A = AC\,\omega_{AB} \quad [\,\underline{\mathbf{i}}\,]$$

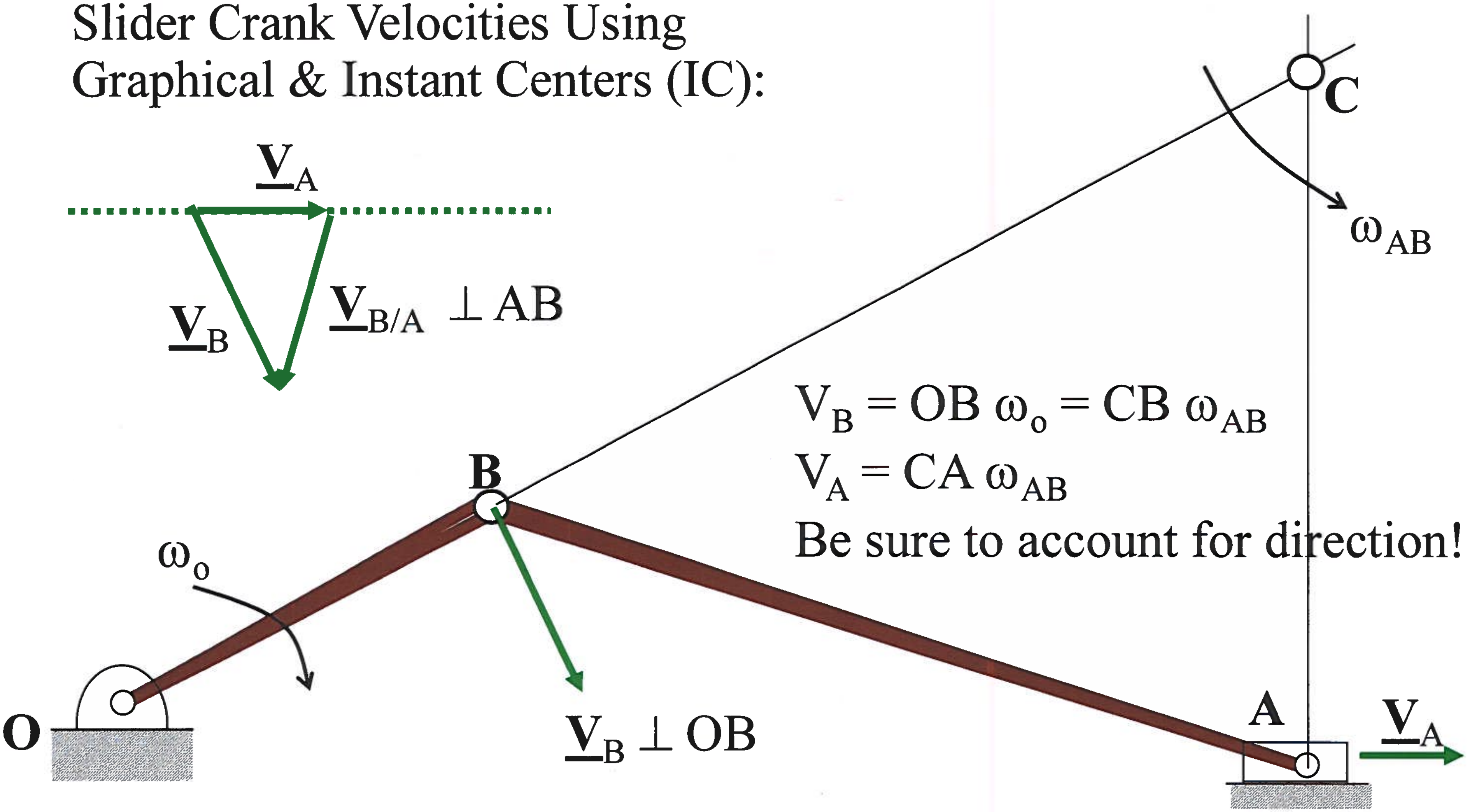
$$V_B = BC\,\omega_{AB} \quad [\,-\underline{\mathbf{i}}\,]$$

$$AC = AB\sin\theta$$
$$BC = AB\cos\theta$$

$$\frac{v_A}{v_B} = \frac{AB\,\omega_{AB}\sin\theta}{AB\,\omega_{AB}\cos\theta}$$

$$\frac{v_A}{v_B} = \frac{\sin\theta}{\cos\theta}$$

# Slider Crank Velocities Using Graphical & Instant Centers (IC):




$$V_B = OB\ \omega_o = CB\ \omega_{AB}$$

$$V_A = CA\ \omega_{AB}$$

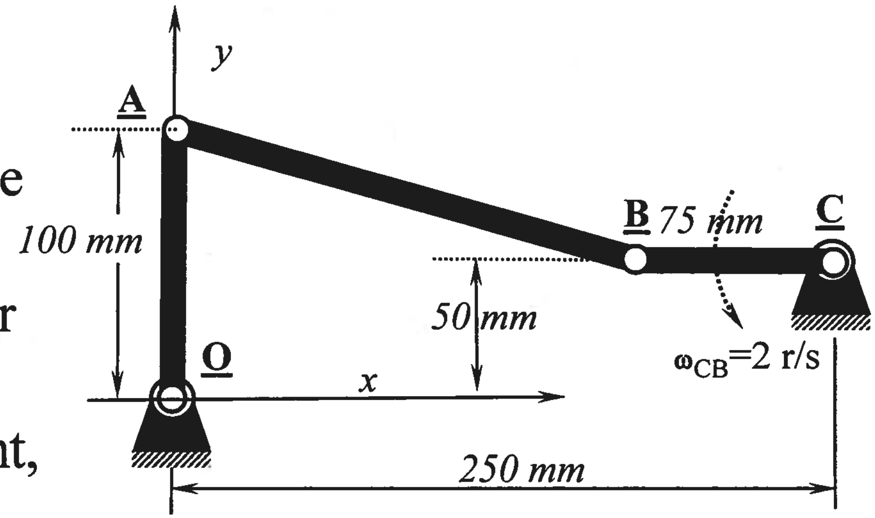
Be sure to account for direction!

$$V_A = (OB / CB)\ CA\ \omega_o$$

# Example: Planar Kinematics of Rigid Bodies *(~Meriam&Kraige Ex 5.8)*

**Given:**

- Crank CB oscillates about C through a limited arc causing rocker OA to oscillate about O. When crank CB reaches horizontal, OA is vertical and the angular velocity of CB is 2 radians per second counterclockwise (CCW). For this instant,




**Find:**

A. The angular velocity of link **AB**

B. The angular velocity of link **OA**

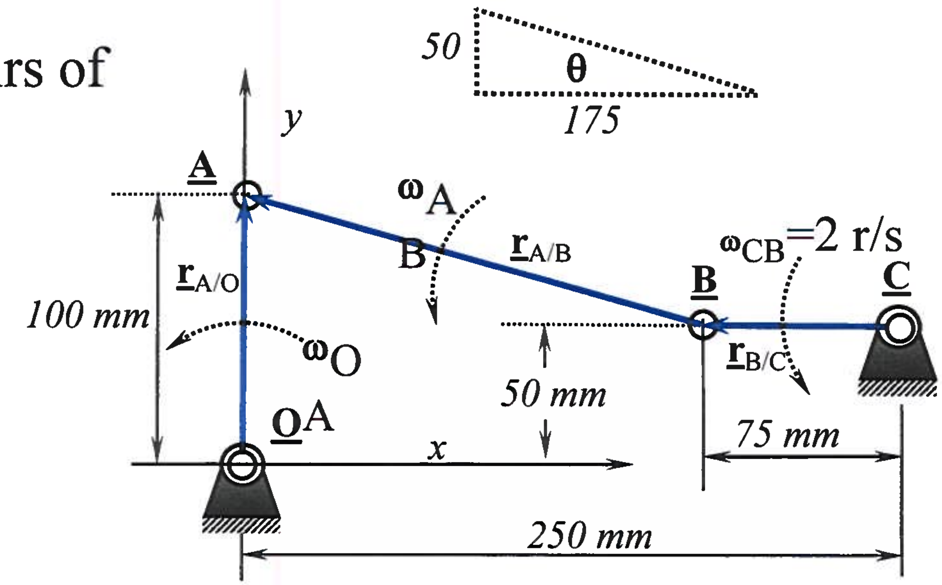
**Solution:**

- Three rigid bodies (links) need kinematics (velocities) to be established
  - **OA** & **CB** pure rotation (1DOF each => $\omega_{OA}$ & $\omega_{CB}$)
  - **AB** exhibits general plane motion (3 DOF)
- Pin joints relate the kinematics (motion) of coincident points on the separate RB's.

# Example: Planar Kinematics of Rigid Bodies *(~Meriam&Kraige Ex 5.8)*

**Solution (cont'd):**

- Relative velocity relationships for pairs of points on the three links




(1) $$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_C + \underline{\mathbf{v}}_{B/C} = \underline{0} + \underline{\omega}_{CB} \times \underline{\mathbf{r}}_{B/C}$$
$$= (2\ r/s)\underline{\mathbf{k}} \times (-75\,mm)\underline{\mathbf{i}}$$
$$= -150\,mm/s\ \underline{\mathbf{j}}$$

(2) $$\underline{\mathbf{v}}_A = \underline{\mathbf{v}}_O + \underline{\mathbf{v}}_{A/O} = \underline{0} + \underline{\omega}_{OA} \times \underline{\mathbf{r}}_{A/O}$$
$$= (\omega_{OA}\ r/s)\underline{\mathbf{k}} \times (100\,mm)\underline{\mathbf{j}}$$
$$= -100\,\omega_{OA}\ mm/s\ \underline{\mathbf{i}}$$

(3) $$\underline{\mathbf{v}}_A = \underline{\mathbf{v}}_B + \underline{\mathbf{v}}_{A/B} = \underline{\mathbf{v}}_B + \underline{\omega}_{AB} \times \underline{\mathbf{r}}_{A/B}$$
$$= -150\,mm/s\,\underline{\mathbf{j}} + (\omega_{AB}\ r/s)\underline{\mathbf{k}} \times \{(75-250\,mm)\underline{\mathbf{i}} + (100-50\,mm)\underline{\mathbf{j}}\}$$
$$= (-175\omega_{AB} - 150)\underline{\mathbf{j}} - 50\,\omega_{AB}\,\underline{\mathbf{i}} \quad (mm/s)$$

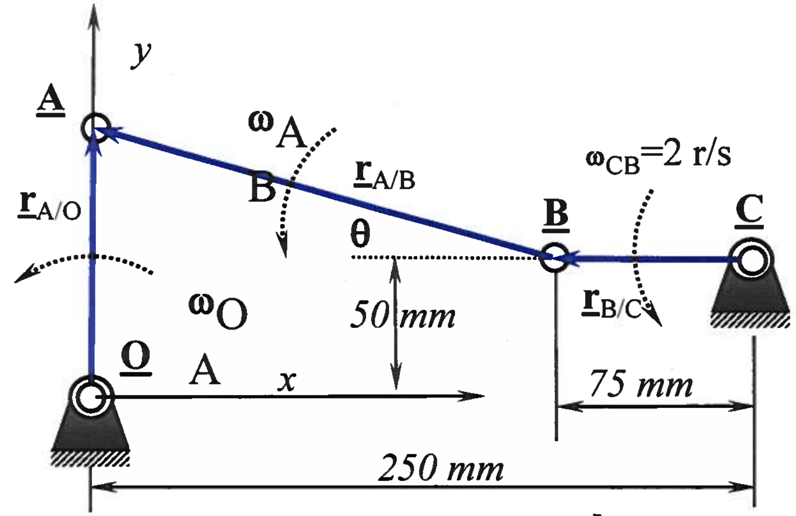
- From (2) & (3), equating **i** & **j** components

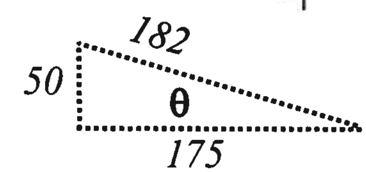
$$\underline{\mathbf{j}} \Rightarrow\ 0 = (-175\omega_{AB} - 150) \quad \Rightarrow \quad \omega_{AB} = -150/175 = -6/7\ (r/s),\ i.e.CW$$

$$\underline{\mathbf{i}} \Rightarrow -100\,\omega_{OA} = -50\,\omega_{AB} \quad \Rightarrow \quad \omega_{OA} = 50/100\,\omega_{AB} = -3/7\ (r/s),\ i.e.CW$$

# Example: Planar Kinematics of Rigid Bodies *(~Meriam&Kraige Ex 5.8)*

## Alternate: Graphical Solution (cont'd):




- Construct **velocity polygon** for the relative velocity constraint

$$\underline{\mathbf{v}}_A = \underline{\mathbf{v}}_B + \underline{\mathbf{v}}_{A/B}$$

D̸M = D̸M̸ D̸M

- As before, $\underline{\mathbf{v}}_B$ easily computed

(1) $\underline{\mathbf{v}}_B = \omega_{CB} r_{B/C} \perp \underline{\mathbf{r}}_{B/C}$

$$= (2\ r/s)(75\,mm)\underline{\mathbf{j}} = -150\,mm/s\ \underline{\mathbf{j}}$$

- $\underline{\mathbf{v}}_{A/B}$ is perpendicular (⊥) to $\underline{\mathbf{r}}_{A/B}$ &

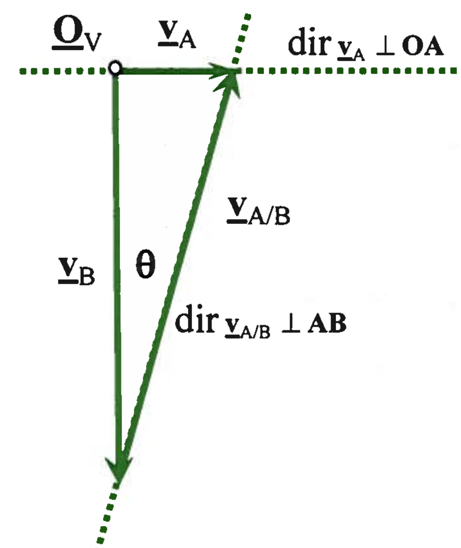
$$\left|\underline{\mathbf{v}}_{A/B}\right| = \omega_{A/B} r_{A/B}$$

- $\underline{\mathbf{v}}_A$ is horizontal (⊥ to $\underline{\mathbf{r}}_{A/O}$)

$$\left|\underline{\mathbf{v}}_A\right| = \omega_{OA} r_{A/O}$$

- Intersection of *lines of action* for $v_A$ & $v_{A/B}$ sets actual sizes for each vector
- Now measure (&/or compute) size of each vector based on scale used for $\underline{\mathbf{v}}_B$

**Velocity Polygon**



# Example: Planar Kinematics of Rigid Bodies
*(~Meriam&Kraige Ex 5.8)*

## **Graphical Solution (cont'd):**

- From the **velocity polygon** geometry $v_A$ and $v_{A/B}$ thus $\omega_{OA}$ and $\omega_{A/B}$ can be found

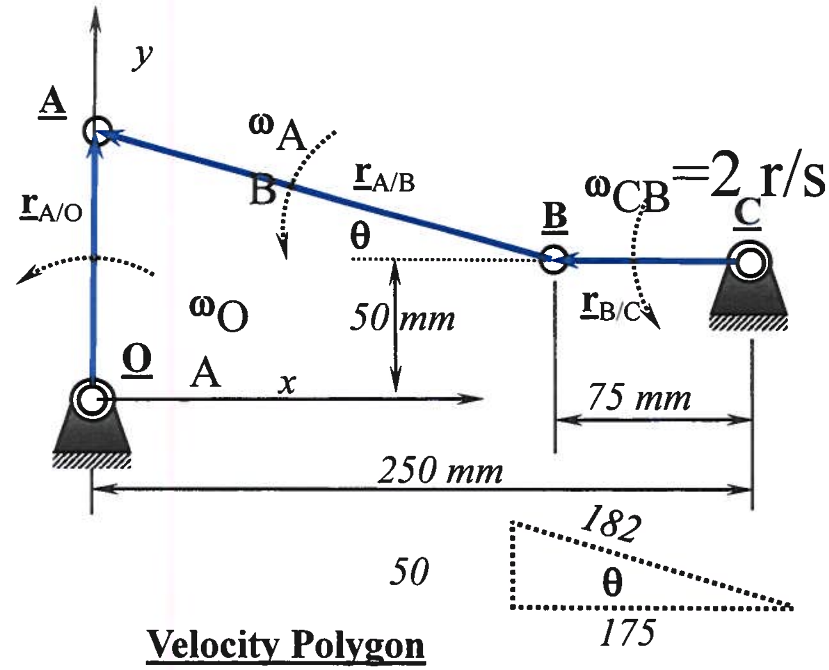
$$|\underline{\mathbf{v}}_A| = |\underline{\mathbf{v}}_B| \tan\theta = 150\frac{50}{175} = 300/7\ (mm/s)$$

$$\Rightarrow \omega_{OA} = \pm\frac{|\underline{\mathbf{v}}_A|}{|\underline{\mathbf{r}}_{AO}|} = \frac{300/7\,(mm/s)}{100\ (mm)} = \underline{\underline{3/7\ (r/s)CW}}$$

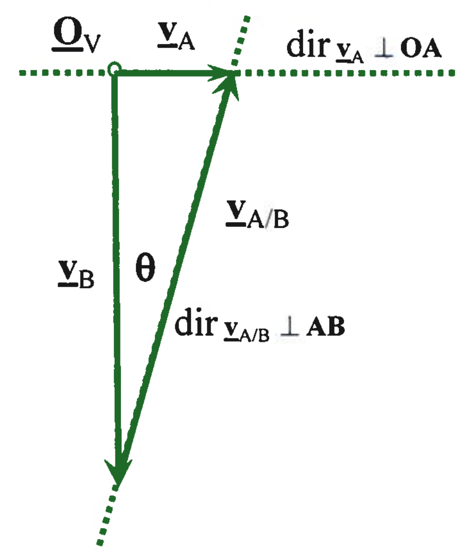
$$|\underline{\mathbf{v}}_{A/B}| = |\underline{\mathbf{v}}_B|/\cos\theta = 150*\frac{182}{175} = 182*6/7\ (mm/s)$$

$$\Rightarrow \omega_{A/B} = \pm\frac{|\underline{\mathbf{v}}_{A/B}|}{|\underline{\mathbf{r}}_{AB}|} = \frac{182*6/7\,(mm/s)}{182\ (mm)} = \underline{\underline{6/7\ (r/s)CW}}$$

- Velocity polygon can be used to quickly validate your answers and/or determine rotation directions




**Velocity Polygon**



# Kinetics Summary

- Three general solution approaches for establishing the governing equations of motion (EOM) => Which one to use?

  i) Newton's Laws

$$\sum \mathbf{F} = m\,\underline{\mathbf{a}}_{CG} \qquad \sum \underline{\mathbf{M}}_{P} = I_{CG}\,\alpha + r_{eff}\, m a_{CG}$$

  ii) Work- Energy & Conservation of Energy

$$U_{A-B} = \int_{s_A}^{s_B} m a_t\, ds = \int_{v_A}^{v_B} m v\, dv = \frac{1}{2} m\left(v_B^2 - v_A^2\right) = \Delta T_{A-B}$$

$$U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}$$

  iii) Impulse - Momentum & Conservation of Momentum

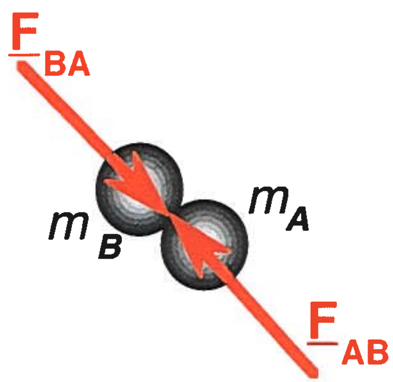
$$\underline{\mathbf{I}} = \int \underline{\mathbf{F}}_R\, dt = \int d\underline{\mathbf{L}} = \Delta \underline{\mathbf{L}}$$

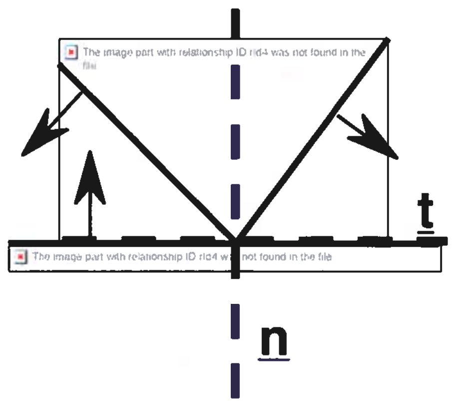
– Typical forces

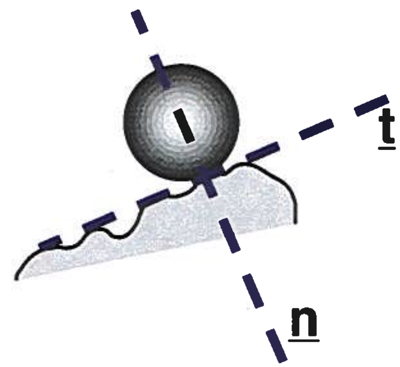
- Springs $F = k\,(s - s_0)$
- Friction $\mathbf{F}_f = \mu_{s/k}\,\mathbf{N}$
- Gravitation $\underline{\mathbf{F}} = m\,\underline{\mathbf{g}}$

# Particle Kinetics: Free Body Diagrams

- <u>Free Body Diagrams</u>:
  - Isolate the particle/system of interest (i.e. boundaries)
  - For noting action-reaction between particles/bodies it is important to identify the **<u>common normal-tangent</u>** **<u>@ the point of contact</u>** (often one or the other is easily identified)



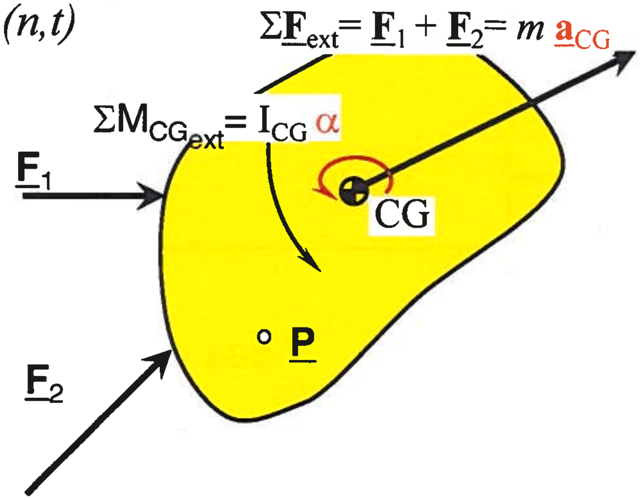

  - Include ALL forces (& later => moments)
    - Field forces (gravity, electro-magnetic fields etc)
    - Viscous forces (aerodynamic drag, fluid flows, etc)
    - Contact forces (touching elements) -- Most common
  - For motion over an interval --- draw in a general position!

# Kinetics of Rigid Bodies – Newtons Law (2D)

$\sum \underline{\mathbf{F}}_{ext} = m\underline{\mathbf{a}}_{CG}$ → 2 Kinetic constraints: $(x,y)$, $(r,\theta)$, $(n,t)$

$\sum \mathbf{M}_{CG_z} = \mathbf{I}_{CG_z}\alpha$ → +1Rot. Kinetic constraint

- 3 Kinetic Constraints per Rigid Body!

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- Alternate Form: for $\Sigma M_P$ where **P ≠ CG**

$$\sum \mathbf{M}_P = \mathbf{I}_{CG}\alpha + (\underline{r}_{G/P} \times m\underline{\mathbf{a}}_{CG})_z$$

$$= \mathbf{I}_{CG}\alpha + ma_{CG}(\pm d_{eff})$$

***OR***

$$\sum \mathbf{M}_P = \mathbf{I}_P\alpha + (\underline{r}_{G/P} \times m\underline{\mathbf{a}}_P)_z$$

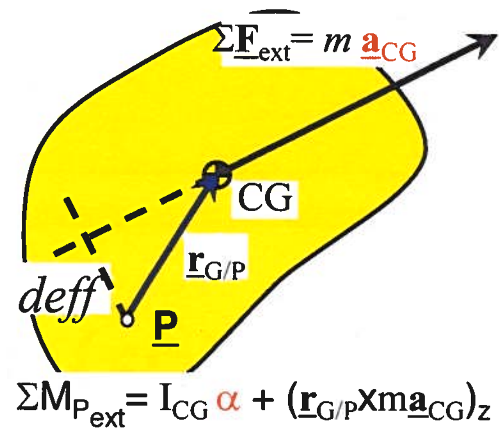
→ IFF **P** is fixed, $\underline{\mathbf{a}}_P = 0$!

$$\sum \mathbf{M}_P = \mathbf{I}_P\alpha$$

$I = mk^2$ => *k rad of gyration*

$\mathrm{I}_{P_{zz}} = \mathrm{I}_{CG_{zz}} + \mathrm{md}^2$ || *axis Xfer*

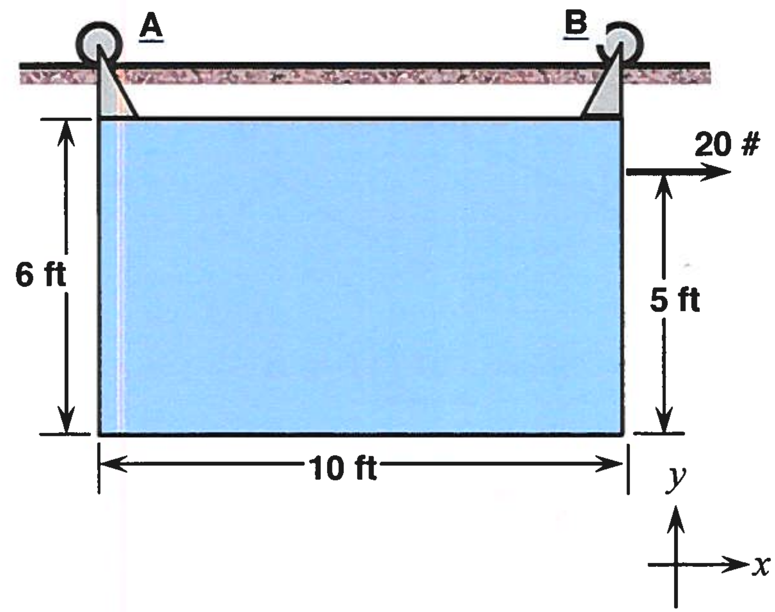
$k = \sqrt{\mathrm{I}/\mathrm{m}}$ => $k_{P_{zz}} = \sqrt{k_{CG}^2 + \mathrm{d}^2}$



# Rigid Body Kinetics – Planar Motion (2D)

## Given:

- A sliding warehouse door rides on ideal rollers & weighs 100#
- Assume the door weight is uniformly distributed

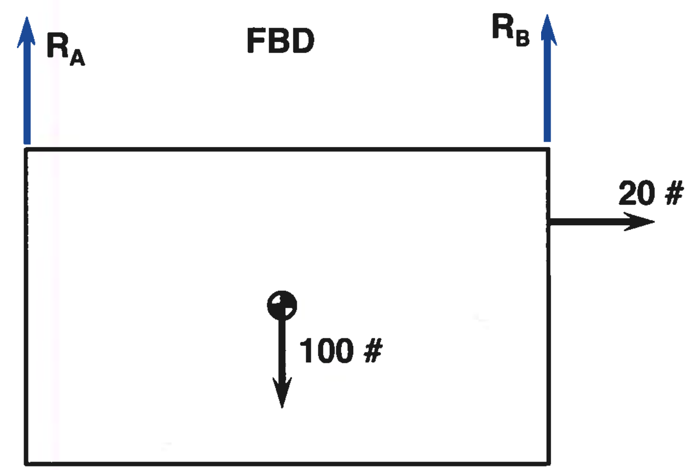



## Find:

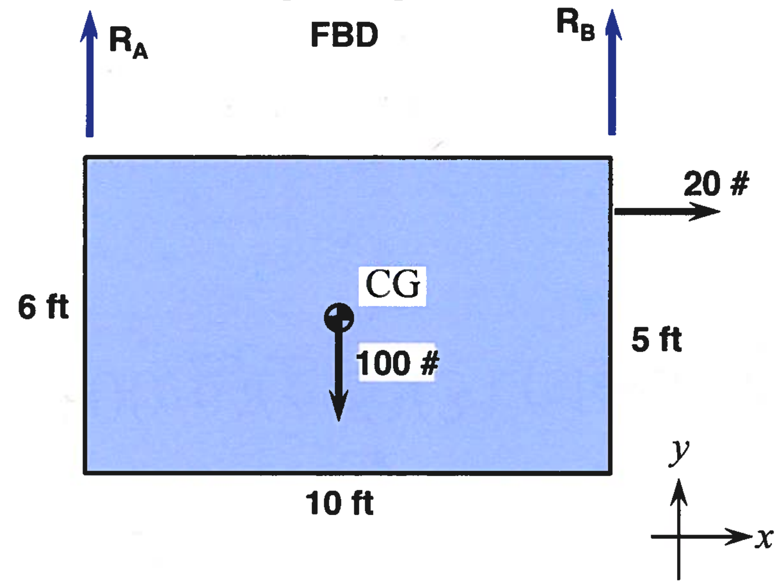
- The reactions at the roller supports
- The acceleration of the door.

## Solution:

- Rectilinear motion: horizontal, no rotation
- IDEAL Rollers: Frictionless, massless
- Construct FBD with reactions properly AT POINT OF CONTACT!



# Rigid Body Kinetics – Planar Motion (2D)




**Solution (continued):**

- Newton's Law (3 kinetic constraints/RB)

$$\Sigma F_x: \; 20\,\# = (100\,\#/g)\, a_{CG_x}$$

$$a_{CG_x} = g/5\; ft/s^2 \quad (g = 32.2\; ft/s^2)$$

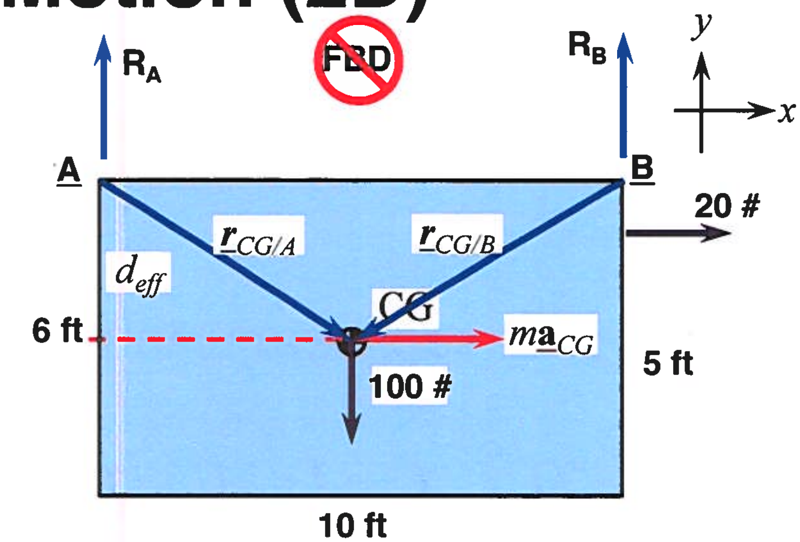
$$\Sigma F_y: \; R_A + R_B - 100\,(\#) = m\, a_{CG_y} = 0\;!$$

$$\Sigma M_{CG}\;(\text{CCW+}): \quad 5\,(R_B - R_A) - 2\,(20)\,(ft\text{-}\#) = I_{CG}\,\alpha = 0\;!$$

- Use last two equations to resolve the two unknown reactions $R_A$ & $R_B$

$$R_A = \underline{\underline{46\,\#}} \qquad\qquad R_B = \underline{\underline{54\,(\#)}}$$

# Rigid Body Kinetics – Planar Motion (2D)




**Alternate Solution (continued):**

- Newton's Law (3 kinetic constraints/RB)

$$\Sigma F_x \Rightarrow \quad a_{CG_x} = g/5\ ft/s^2$$

$$\Sigma F_y \Rightarrow \quad R_A + R_B = 100\ (\#)$$

- Sum moments about a point other than CG

$$\Sigma M_P = I_{CG}\,\alpha + (\underline{\mathbf{r}}_{CG/P} \times m\underline{\mathbf{a}}_{CG})_z \quad \& \quad \alpha = 0!$$

$$\Sigma M_A\text{: } 10\,R_B + 1{*}20 - 100{*}5\ (ft\text{-}\#) = (100/g)(g/5)(3)\ (\text{slg-ft}^2/\text{s}^2)$$

$$\Rightarrow R_B = \underline{\underline{54\ (\#)}}$$

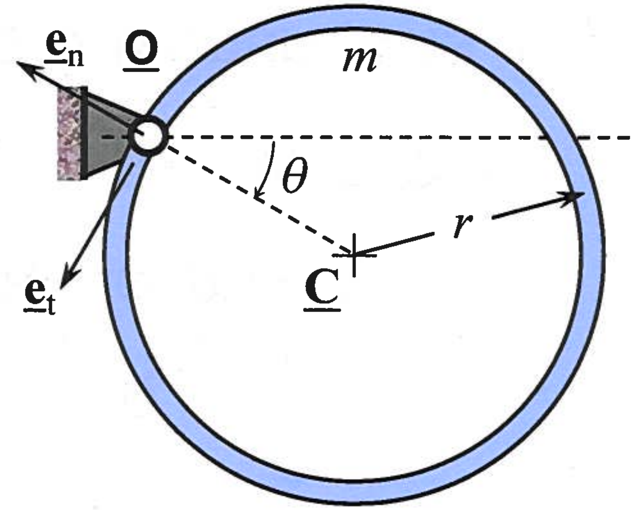
$$\Sigma M_B\text{: } -10\,R_A + 1{*}20 + 100{*}5\ (ft\text{-}\#) = (100/g)(g/5)(3)\ (\text{slg-ft}^2/\text{s}^2)$$

$$\Rightarrow R_A = \underline{\underline{46\ \#}}$$

# Rigid Body Kinetics – Planar Motion (2D)

**Given:**

- A thin ring of mass $m$ is free to rotate in the vertical plane about the frictionless pin joint at ***O***.
- Its angular velocity is $\omega_0$ (CW) when $\theta$=0°



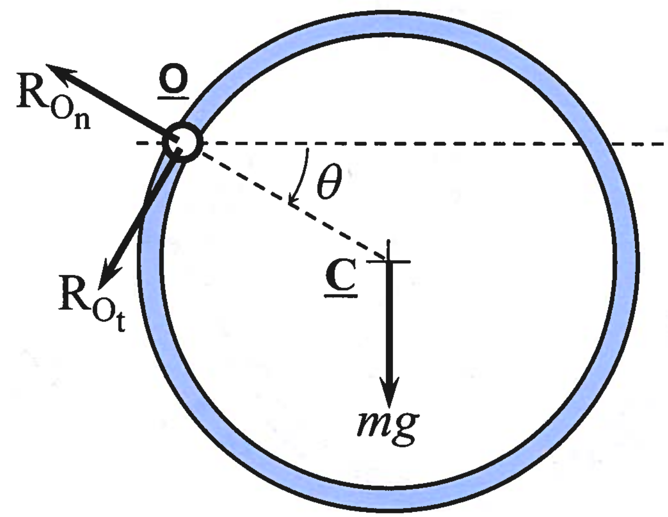

**Find:** *(for any arbitrary angle $\theta$)*

- The reactions forces at **O**
- The angular velocity of the ring

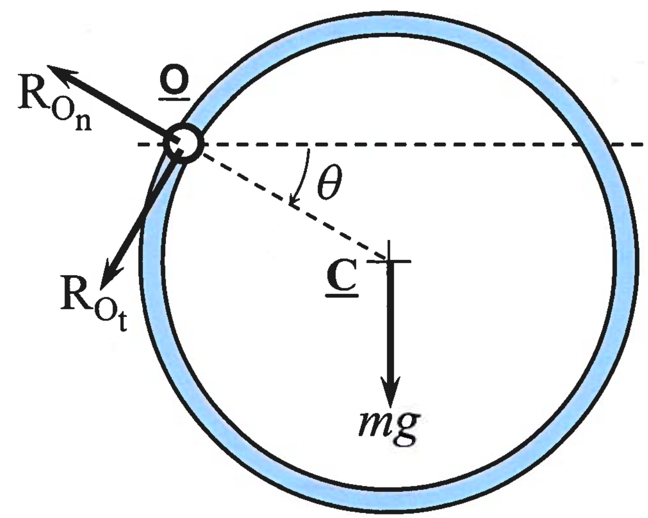
**Solution:**

- Fixed axis rotation about **O**
- Frictionless pin joint
- Construct FBD using $n$-$t$ axes (+z into page – CW +)

***FBD***



# Rigid Body Kinetics – Planar Motion (2D)




**Solution (continued):**

- Newton's Law (3 kinetic constraints/RB)

$$\Sigma F_n:\ R_{O_n} - mg\sin\theta = m\,a_{C_n}$$

$$\Sigma F_t:\ R_{O_t} + mg\cos\theta = m\,a_{C_t}$$

$$\Sigma M_C \text{ (CW+)}:\ -R_{O_t}\,r = I_C\,\alpha$$

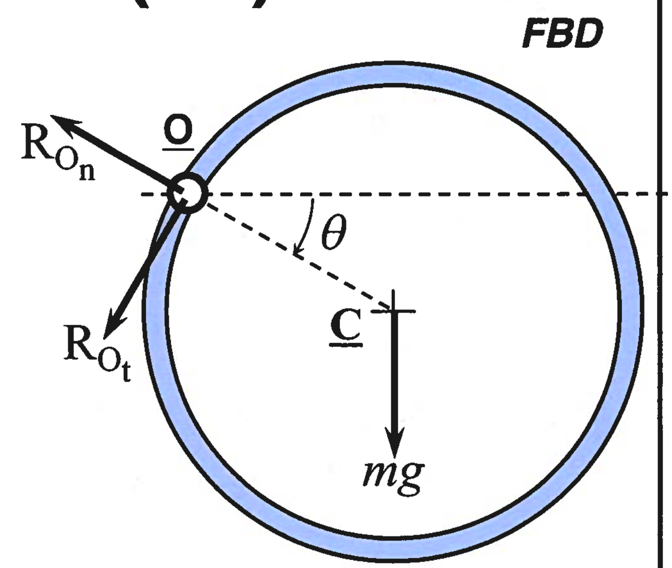
$$I_C = mr^2$$

- 3 kinetic constraints & 5 unknowns: $R_{o_n}$, $R_{o_t}$, $a_{C_n}$, $a_{C_t}$, $\alpha$
- Look to ***kinematics*** to provide necessary constraints!
  - Fixed axis rotation => $a_{C_n} = \omega^2 r$ & $a_{C_t} = \alpha\, r$
- Now 3 kinetic + 2 kinematic constraints & 6 unknowns ($\omega$)!

$\alpha$ <= *derivative* $\omega$ 6 equations ⇔ 6 unknowns C.B.S.!

# Rigid Body Kinetics – Planar Motion (2D)

**Solution (continued):**




$I_C = mr^2$

- Combine $\Sigma M$ & $I_C$

$$-R_{O_t}\, r = I_C\, \alpha \Rightarrow \alpha = -R_{O_t}\, r/(mr^2)$$

$$\alpha = -R_{O_t}/(mr)$$

- Combine $\Sigma F_t$ & $a_{C_t} = \alpha\, r$

$$R_{O_t} + mg\cos\theta = m\,\alpha\, r$$

- Sub for $\alpha$ & resolve $R_{O_t}$

$$R_{O_t} + mg\cos\theta = m\, r\, [-R_{O_t}/(mr)] \quad \Rightarrow \quad R_{O_t} = \underline{\underline{-\, mg\cos\theta / 2}}$$

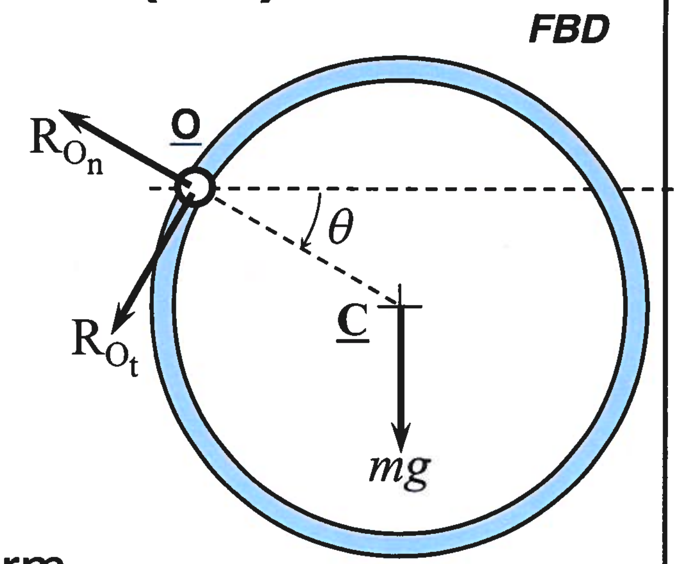
- Now $\alpha$ can be determined

$$\alpha = -\,(-\, mg\cos\theta / 2)/(mr) \quad \Rightarrow \quad \alpha = \underline{\underline{g\cos\theta / (2r)}}$$

- Remaining unknowns: $R_{o_n}$, $a_{C_n}$, $\omega$ => Now what?

# Rigid Body Kinetics – Planar Motion (2D)

## Solution (continued):




$I_C = mr^2$

- Knowing $\alpha = g\cos\theta / 2r$
  - Integrate to get $\omega = f_2(\theta)$
  - Use $\omega$ to get $a_{C_n} = \omega^2 r$
  - Use $a_{C_n}$ & $\Sigma F_n$ to get $R_{O_n}$
- Variables $(\alpha, \omega, \theta)$, no $t$ => use $\alpha d\theta = \omega d\omega$ form

$$\int_0^\theta \frac{g}{2r}\cos\theta\, d\theta = \int_{\omega_0}^{\omega_\theta} \omega\, d\omega$$

$$\Rightarrow \frac{g}{2r}\sin\theta\Big|_0^\theta = \frac{1}{2}\omega^2\Big|_{\omega_0}^{\omega_\theta} \Rightarrow \omega_\theta^2 = \omega_0^2 + \frac{g}{r}\sin\theta$$

$$a_{C_n} = \omega_\theta^2 r = (\omega_0^2 + \frac{g}{r}\sin\theta)r = r\omega_0^2 + g\sin\theta$$

$$R_{O_n} - mg\sin\theta = m(r\omega_0^2 + g\sin\theta)$$

$$R_{O_n} = \underline{\underline{mr\omega_0^2 + 2mg\sin\theta}}$$

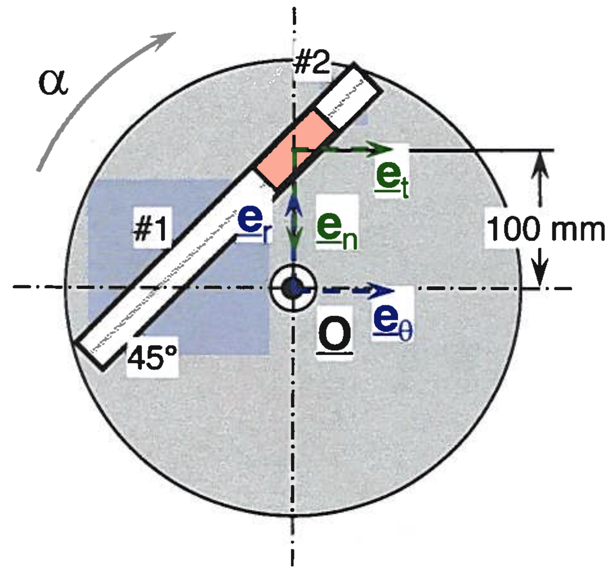
- Note $\Sigma M_O = I_O\,\alpha$ & eliminates reactions!

$I_O = mr^2 + mr^2$

$mgr\cos\theta = 2mr^2\,\alpha$

$\alpha = g\cos\theta/(2r)$

# Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74




**Given**:

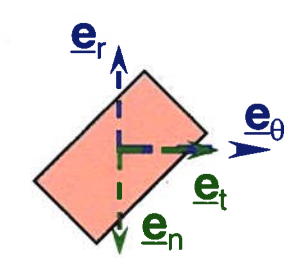
- The slider ($m$=2 kg) fits loosely in the smooth slot of the disk which lies in a horizontal plane and rotates about a vertical axis through point **O**.
- The slider is free to move only slightly along the slot in either direction before one (but not both) of the two wires #1 or #2 becomes taut.
- The disk starts from rest at time $t = 0$ and has a constant clockwise angular acceleration of $\alpha$=0.5 r/s$^2$.

**Find**:

(A) Determine the **TENSION** (**$T_2$**) in wire #2 at $t$ =1 second

(B) Determine the **REACTION FORCE** (**N**) between the slot and the block, again at $t$ =1 second.

(C) Determine the **TIME** ($t$) at which the tension in wire #2 goes slack and wire #1 becomes taut.




**Solution**:

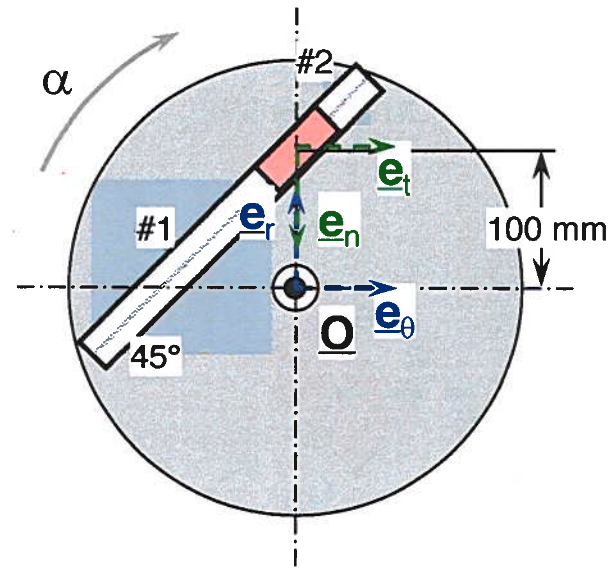
- Asks for FORCES (**T**,**N**) so we must first establish kinematics (accelerations!)
- "Move only slightly" means it is effectively fixed relative to the slot/disk, thus
- The slider travels a circle about **O** & path ($\mathbf{e}_n$-$\mathbf{e}_t$) axes

  or polar ($\mathbf{e}_r$-$\mathbf{e}_\theta$) axes are convenient

# Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74

**Solution (continued)**:

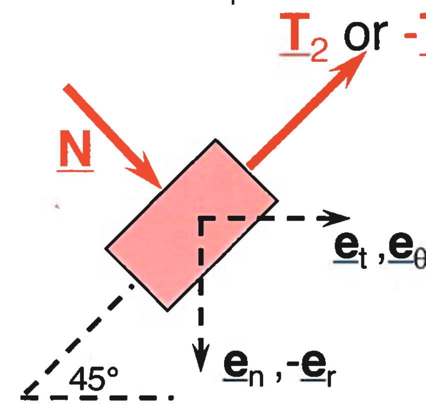
- Construct FBD
- Use disk kinematics ($\alpha$=0.5 r/s$^2$ CW constant) to determine slider's total acceleration




$$\rho = 0.100m = r \quad \Rightarrow constant$$

$$\therefore \quad \dot{\rho} = \ddot{\rho} = \dot{r} = \ddot{r} = 0$$

- Not instantaneous - integrate angular acceleration




$$\int_0^{\omega} d\omega = \int_0^{t} \alpha\, dt = \int_0^{t} 0.5\, dt$$

$$\omega = 0.5t$$

$$\underline{\mathbf{a}}_s = \alpha\, r\, \underline{\mathbf{e}}_\theta - r\omega^2\, \underline{\mathbf{e}}_r$$

$$= \dot{v}\, \underline{\mathbf{e}}_t + \frac{v^2}{\rho} \underline{\mathbf{e}}_n = \alpha\, r\, \underline{\mathbf{e}}_t + \omega^2\, r\, \underline{\mathbf{e}}_n$$

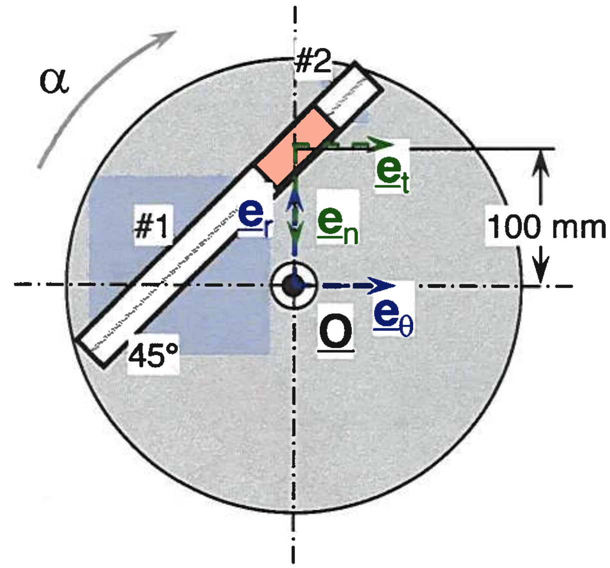
# Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74

**Solution (continued)**:

- Newton's Law can be applied along ANY two independent directions to resolve unknown reactions
  - Sum force components along $(n\text{-}t,\ r\text{-}\theta)$

$$\mathbf{T}_2 \cos 45 + \mathbf{N} \sin 45 = m\alpha r$$

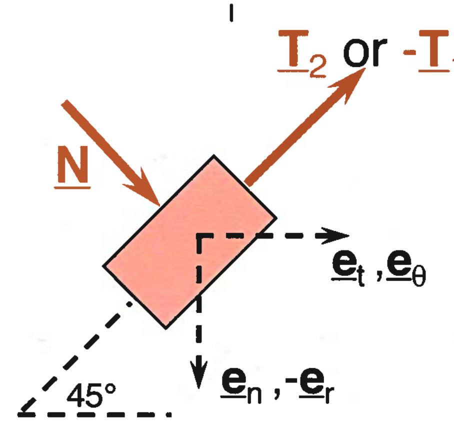
$$\mathbf{T}_2 \sin 45 - \mathbf{N} \cos 45 = -\, m\omega^2 r$$




  - OR to simplify algebra of unknowns, choose the directions along the unknown reactions and sum both *forces* and *acceleration components*

$$\mathbf{T} = m\left(\alpha r \cos 45 - \omega^2 r \sin 45\right) = \frac{mr\sqrt{2}}{2}\left(\alpha - \omega^2\right)$$

$$\mathbf{N} = m\left(\alpha r \cos 45 + \omega^2 r \sin 45\right) = \frac{mr\sqrt{2}}{2}\left(\alpha + \omega^2\right)$$




  - *ASIDE*: This IS the geometric equivalent to simultaneously solving the first set of constraints to yield expressions for the unknowns
  - Noting the similarity of the expressions (± : + for **N**, - for **T**)

$$\mathbf{N}, \mathbf{T}_2 = \frac{mr\sqrt{2}}{2}\left(\alpha \pm \omega^2\right)$$

# Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74

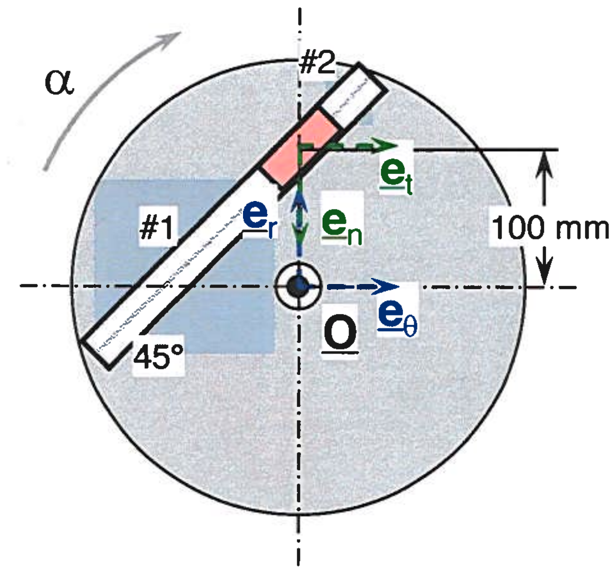
**Solution (continued)**:

- Substituting the known expressions for α & ω (t)

$$\mathbf{N},\mathbf{T}_2 = \frac{mr\sqrt{2}}{2}\left(\alpha \pm \omega^2\right)$$

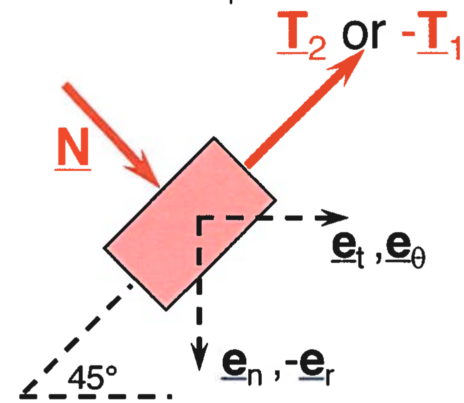
$$= \frac{2\,kg * 0.1\,m * \sqrt{2}}{2}\left\{\,0.5 \pm (0.5\,t)^2\right\}(r/s^2)$$

$$\mathbf{N},\mathbf{T}_2 = \frac{\sqrt{2}}{20}\left\{\,1 \pm 0.5\,t^2\right\}(N)$$




(A) So for $t$=1, the TENSION $\mathbf{T}_2$ is

$$\mathbf{T}_2 = \frac{\sqrt{2}}{20}\left\{\,1 - 0.5(1)^2\right\}\,(N) = \frac{\sqrt{2}}{40}(N) = 0.035(N)$$




(B) At $t$=1, the NORMAL REACTION $\mathbf{N}$ is

$$\mathbf{N} = \frac{\sqrt{2}}{20}\left\{\,1 + 0.5(1)^2\right\}\,(N) = \frac{3\sqrt{2}}{40}(N) = 0.106(N)$$

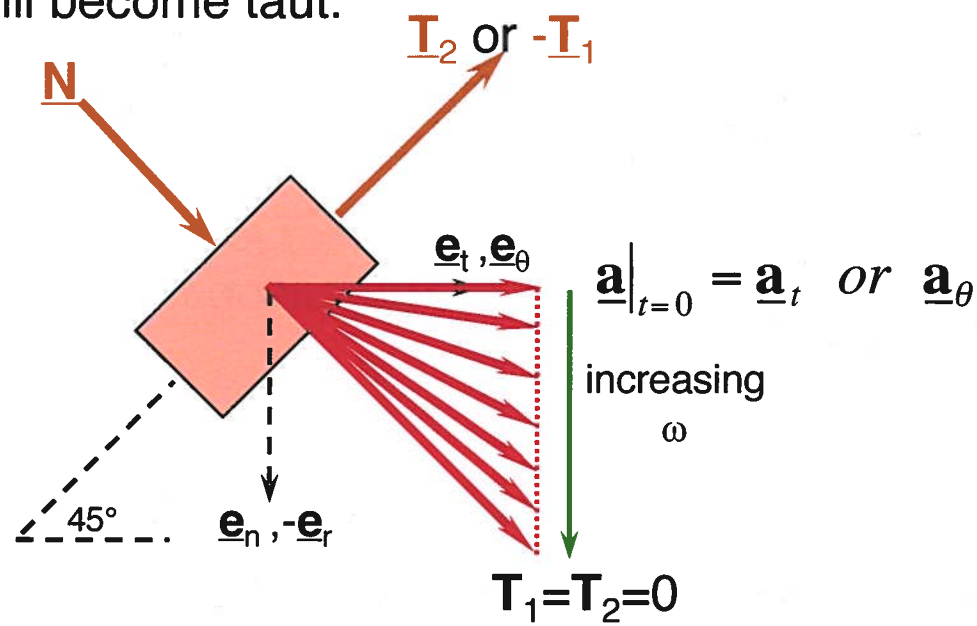
(C) The time when TENSION $\mathbf{T}_2$ goes to zero is

$$\mathbf{T}_2 = \frac{\sqrt{2}}{20}\left\{\,1 - 0.5t^2\right\}\,(N) = 0 \Rightarrow 1 - 0.5t^2 = 0 \Rightarrow t = \sqrt{2} \Rightarrow t = 1.414\,(s)$$

# Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74

**Langiappe**:

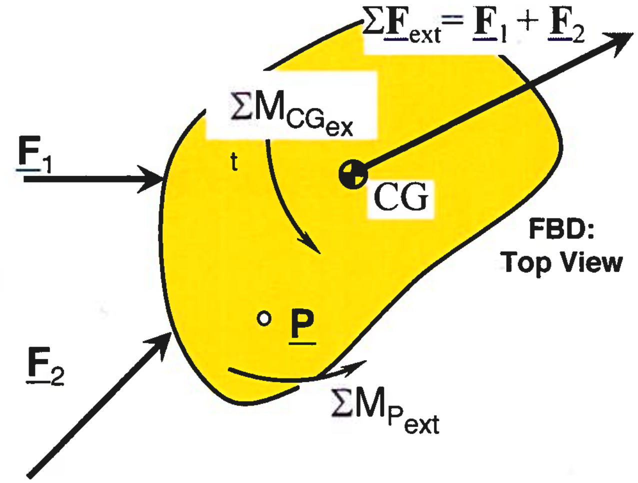
- The acceleration vector starts off completely in the lateral ($\theta$ or *t*) direction here ($\omega$=0). Since cables/wires/ropes cannot PUSH, only $\mathbf{T}_2$ can be engaged in balancing the (*r* or *n*) component of the side wall reaction **N**
- The tangential acceleration component remains constant
- As the disk speeds up ($\omega$ >0), the normal component increases
- When the total acceleration vector aligns with the normal reaction force between the block & slot, the cord/wire tensions are both zero momentarily, and as $\mathbf{T}_2$ goes slack, $\mathbf{T}_1$ will become taut.



# Kinetics of Rigid Bodies (2D): Impulse-Momentum

- Motion studies: ***Forces/Moments, Velocities (linear/angular), Time***
  - Linear Momentum (*Vector constraint 2D*)

$$\underline{\mathbf{I}}_{ext} = \int \underline{\mathbf{F}}_{ext}\, dt = \Delta m \underline{\mathbf{v}}_{CG} = \Delta \underline{\mathbf{L}}_{CG}$$




  - Angular Momentum (+1 constraint)

    Add RB ROTATION to *Moment of $\underline{\mathbf{L}}_{CG}$*

$$AI_P = \int \mathbf{M}_P\, dt$$

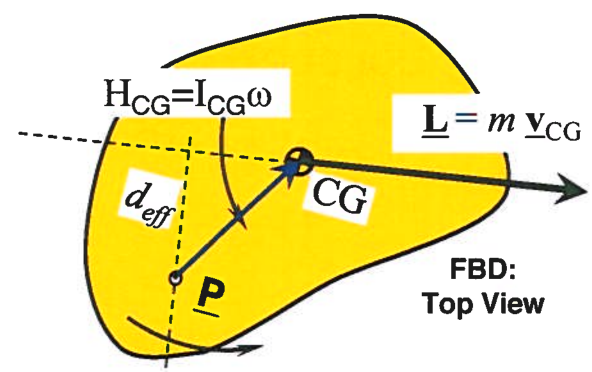
$$= I_{CG}\Delta\omega + (\underline{r}_{G/P} \times m\Delta\underline{\mathbf{v}}_{CG})_z$$

$$= I_P\Delta\omega + (\underline{r}_{G/P} \times m\Delta\underline{\mathbf{v}}_P)_z$$

→ If **P** is **CG** or a fixed point in space

$$AI_{CG_z} = \int \mathbf{M}_{CG_z}\, dt = I_{CG_z}\Delta\omega = \Delta H_{CG}$$

$$AI_{P_z} = \int \mathbf{M}_{P_z}\, dt = I_{P_z}\Delta\omega = \Delta H_P$$




$H_P = I_{CG}\omega + \underline{\mathbf{r}}_{G/P} \times m\, \underline{\mathbf{v}}_{CG}$

$H_P = I_{CG}\omega + m v_{CG} d_{eff}$

*Impact:* ***Coefficent of Restitution***

*Complicated phenomenon with limited applicability*

$$e = \left.\frac{(V_{rel-Sep})}{(V_{rel-App})}\right|_{Common\ Normal}$$

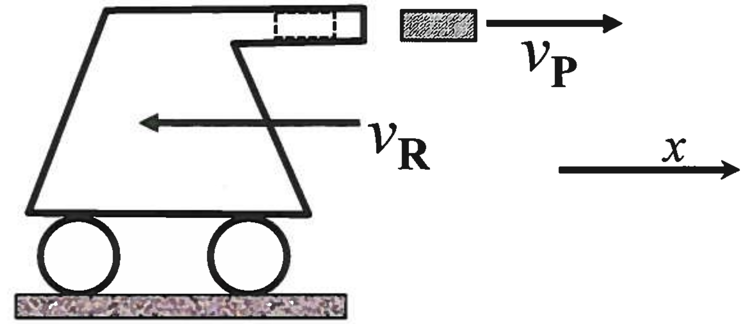
# Example: Conservation of Momentum

**Given**:

- An artillery gun ($m_G$) resting on the ground, fires a shell ($m_P$) with a speed $v_p$

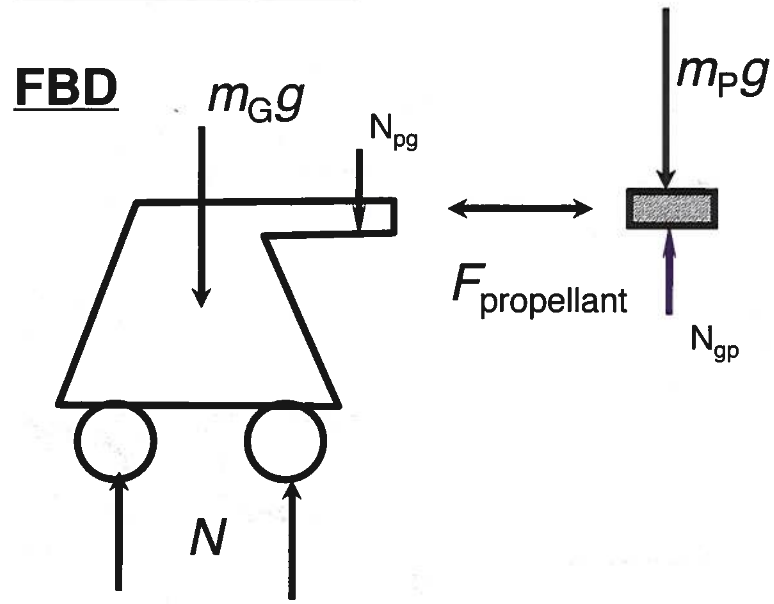
**Find**:

(A) The recoil speed ($v_R$) of the gun




**Solution**:

- Rectilinear motion (i.e. only horizontal motion of interest here)
- FBD of system components, up through the shell leaving the gun barrel
- Propellant firing is internal to the system
  - System momentum is conserved in the horizontal direction




$$\Delta \mathbf{L}_{\mathbf{sys}-x} = 0$$

$$\Delta L_{sys-x} = m_G(v_G - 0) + m_P(v_P - 0) = 0$$

$$v_R = -v_G = \frac{m_P}{m_G} v_p$$

# Example: Conservation of Momentum

**Given**:

- More often, a "muzzle velocity" ($v_{P/G}$)or speed of the shell relative to the gun barrel is specified

**Find**:

(A) The recoil speed ($v_R$) of the gun

**Solution**:

- FBD (same), rectilinear motion (same) & propellant firing is internal (same)

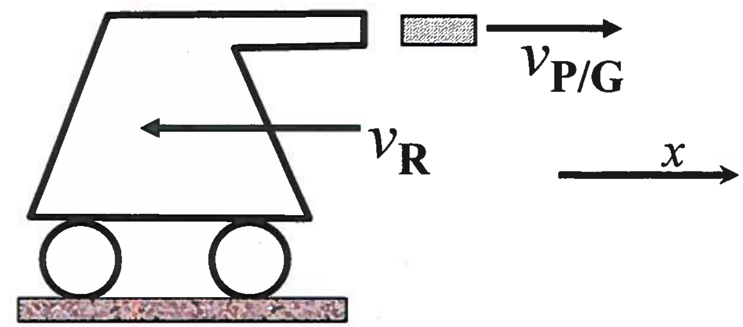
$$\Delta \mathbf{L}_{\mathbf{sys}} = 0$$

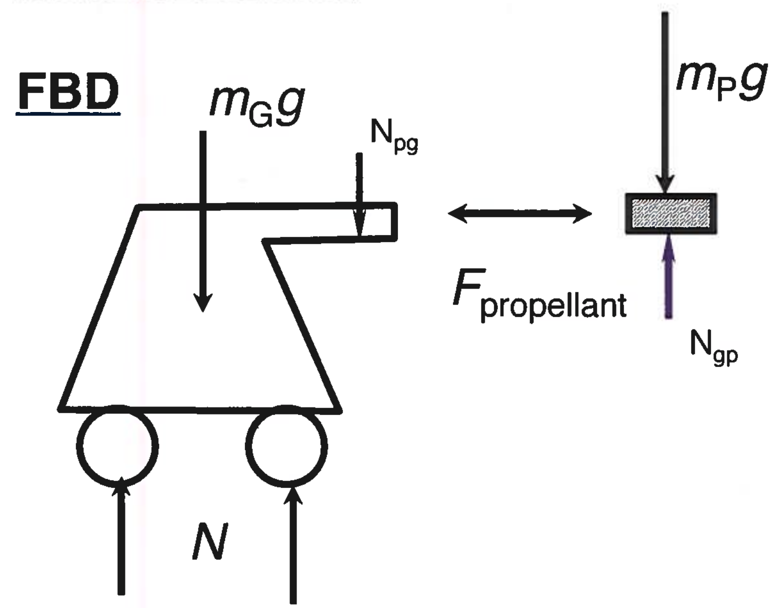
$$\Delta L_{sys-x} = m_G(v_G - 0) + m_P(v_P - 0) = 0$$

$$v_P = v_G + v_{P/G}$$

$$m_G v_G + m_P(v_G + v_{P/G}) = 0$$

$$v_R = -v_G = \left(\frac{m_P}{m_G + m_P}\right) v_{P/G}$$

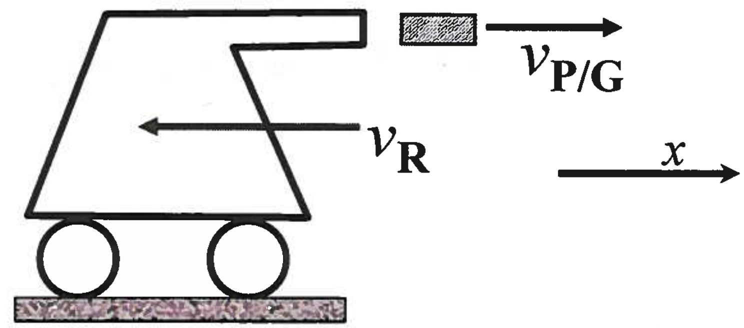


# Example: continued

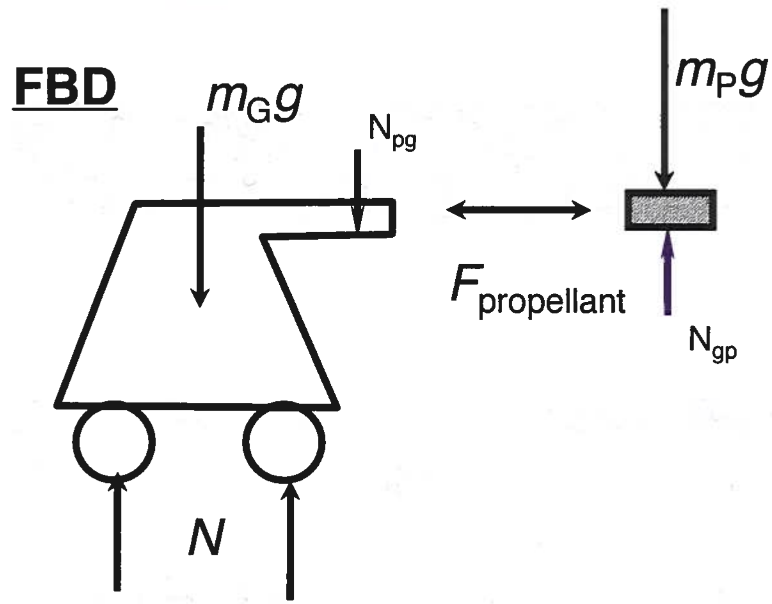
**Asking for more**:

- If resultant muzzle blast occurs over a short time $t_{blast}$, what resultant "kick" is felt by the cannon?




**Solution**:

- An *average* $F_{prop\text{-}avg}$ can be computed to approximate the kick.




$$\Delta \mathbf{L}_{sys} = 0 \qquad \Delta \mathbf{L}_{gun-x} \neq 0$$

$$\Delta L_{gun-x} = I_x = \int -F_{propellant} dt$$

$$= -F_{prop-avg} \int_0^{t_{blast}} dt = -F_{prop-avg} t_{blast}$$

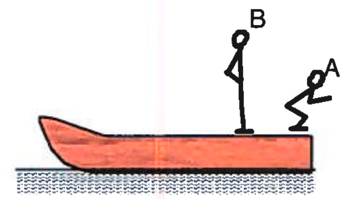
$$F_{prop-avg} = \frac{-1}{t_{blast}} \Delta L_{gun-x} = \frac{-m_G}{t_{blast}}(-v_R - 0) = m_G \frac{v_R}{t_{blast}}$$

$$F_{prop-avg} = m_G \left( \frac{m_P}{m_G} \right) \frac{v_p}{t_{blast}} = \underline{\underline{m_p \frac{v_P}{t_{blast}}}} \quad or \quad F_{prop-avg} = m_G \left( \frac{m_P}{m_G + m_P} \right) \frac{v_{P/G}}{t_{blast}} = \underline{\underline{\left( \frac{m_G m_P}{m_G + m_P} \right) \frac{v_{P/G}}{t_{blast}}}}$$

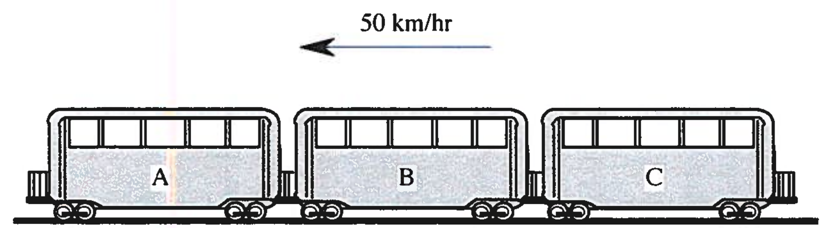
# Example: Conservation of Momentum

**Given**:

- Numerous examples with similar circumstances, rephrasing the wording
  - Kid(s) on a boat in still water, one jumps off
  - Car lands on a barge & skids to rest relative to barge
  - Rail cars collide & stay attached








**Find**:

(A) The resulting speeds of each element

(B) A time it takes to "skid to rest"

**Solution**:

- Similar conservation of momentum relations

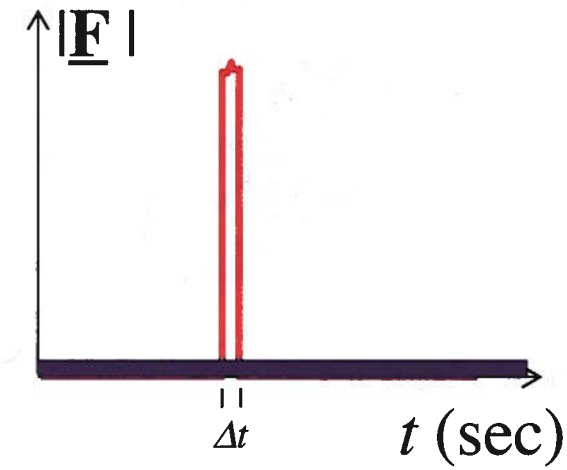
$$\Delta \mathbf{L}_{\mathbf{sys}-x} = 0$$ *=> Resolve velocities*

$$\Delta \mathbf{L}_{components-x} \neq 0$$ *=> Velocities known -> Resolve Net Impulse*

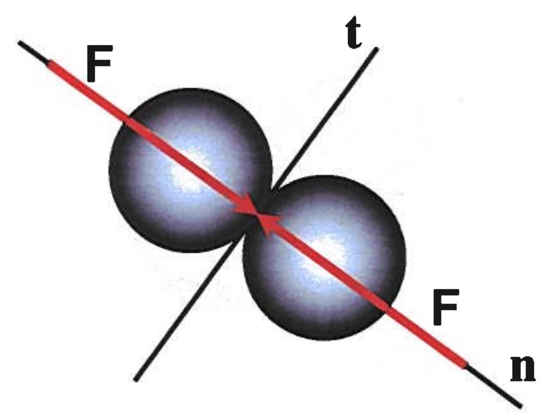
$$\underline{\mathbf{I}} = \int \underline{\mathbf{F}}_R dt = \int d\underline{\mathbf{L}} = \Delta \underline{\mathbf{L}} \quad \Rightarrow \underline{\mathbf{I}} = \underline{\mathbf{F}}_{R-avg} \Delta t = \Delta \underline{\mathbf{L}}$$

# Particle Kinetics: Impulse-Momentum

- Impact Problems:
  - Reformulation of one type of Impulse-Momentum $\underline{\mathbf{I}} = \Delta\underline{\mathbf{L}} = m\Delta\underline{\mathbf{v}}$
  - Impact Forces ($\underline{\mathbf{F}}$) characterized by
    - LARGE MAGNITUDE
    - SHORT TIME DURATION
    - Ex: explosions, collisions, ball-bat, club-golf ball



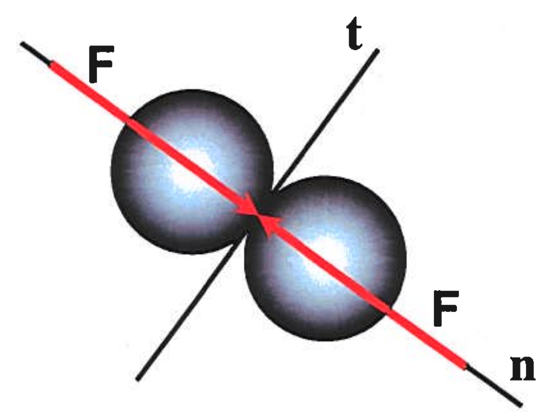

  - Neglect other conventional forces of lesser effect for the short time interval considered as their total effect is negligible
    - Springs
    - Gravity
    - Many Reaction forces (BUT NOT ALL!)




  - Good opportunity to look at the SYSTEM of particles in simplifying the problem (reactions are internal!)

# Particle Kinetics: Impulse-Momentum/ Impact

- Impact
  - Locate Common Normal/Tangent
    - Line of contact/impact - the NORMAL!
  - Forces (**F**) of interaction
    - Equal, Opposite, Co-linear

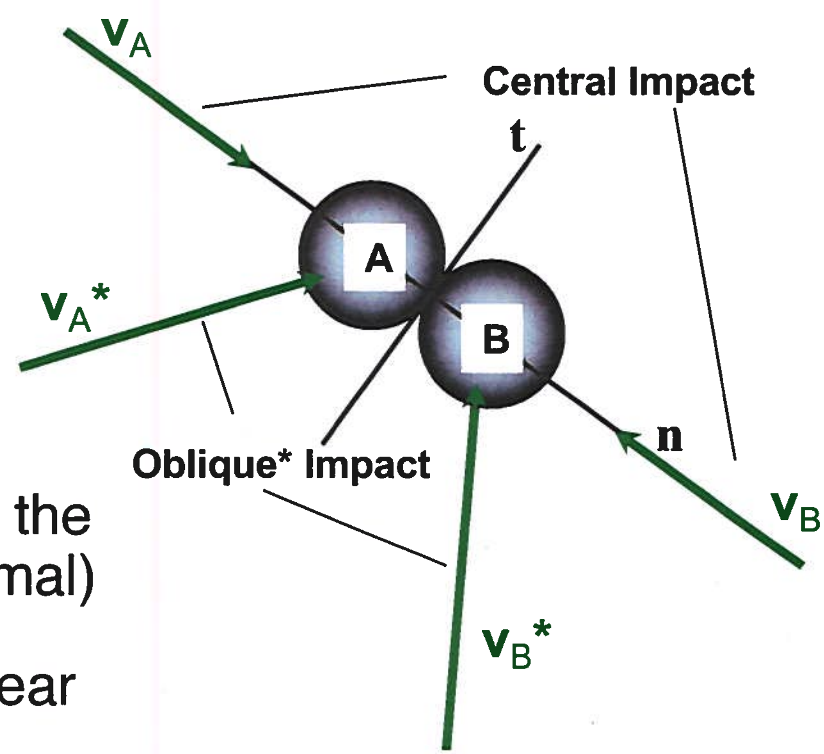



  - Very complex internal phenomena, captured by *Coefficient of Restitution*

$$e = \left.\frac{\left(V_{Relative-Separation}\right)}{\left(V_{Relative-Approach}\right)}\right|_{Common\ \ Normal}$$

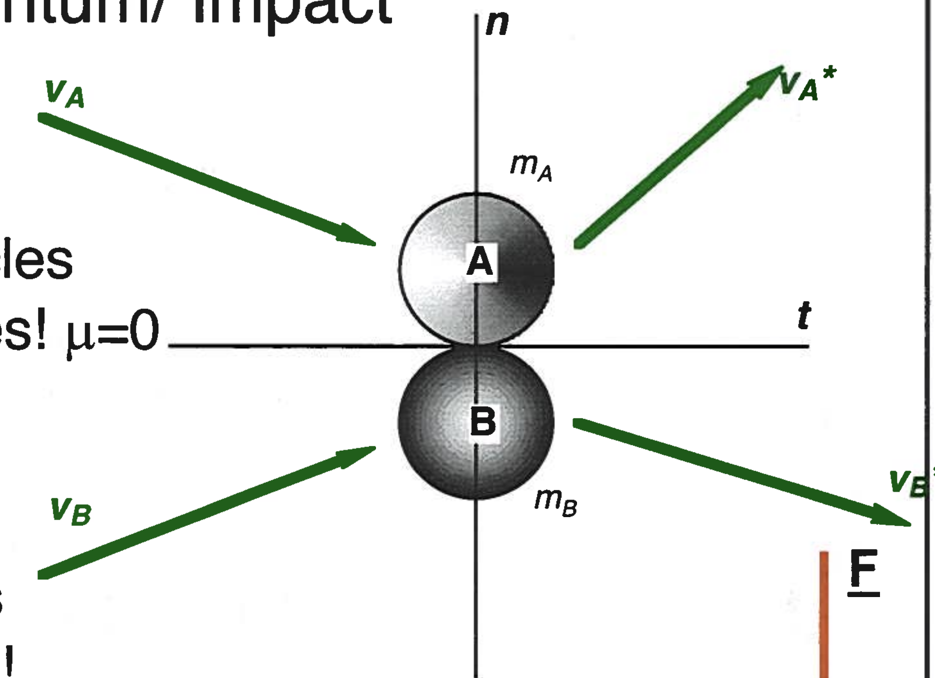
(good derivation in text --- READ IT!)

  - *Central & Oblique Impacts*
    - *Central: Velocities COLINEAR* with the line of impact (i.e. the common normal)
    - *Oblique**: Velocities are NOT co-linear



# Particle Kinetics: Impulse-Momentum/ Impact

- Solving Impact Problems !




(1) Tangential direction: ***individual*** particles have no net external impulsive forces! $\mu=0$

$$m_A v_{At} = m_A v_{At}^* \quad \& \quad m_B v_{Bt} = m_B v_{Bt}^*$$

(2) Normal direction: ***system*** of particles has no net external impulsive forces!

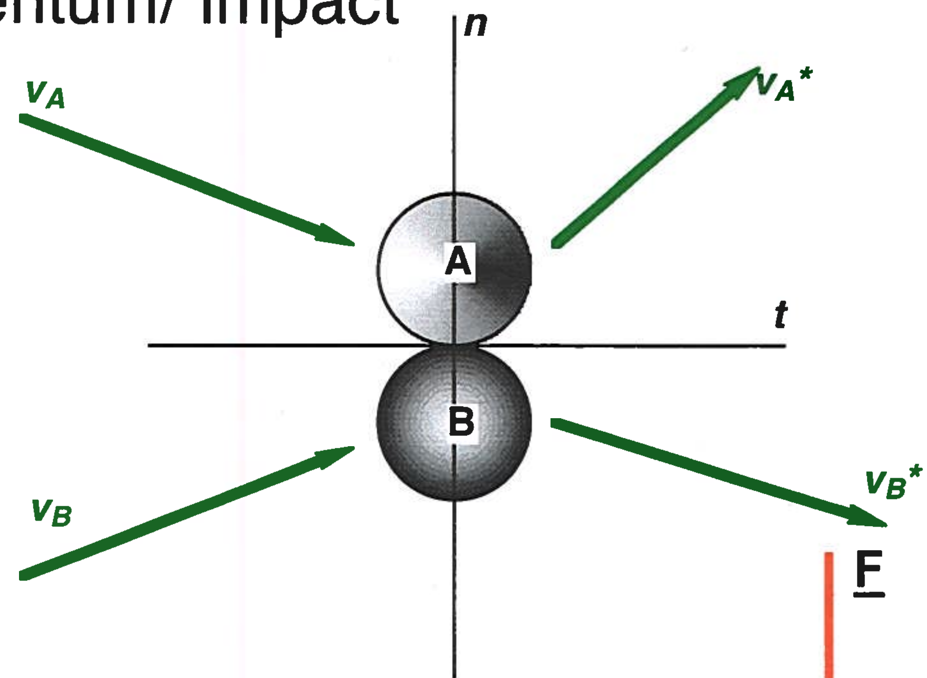

$$\Delta \underline{\mathbf{L}}_{SYS}\Big|_n = 0 \Rightarrow m_A v_{An} + m_B v_{Bn} \quad = \quad m_A v_{An}^* + m_B v_{Bn}^*$$

(3) Coefficient of Restitution: Rel. Velocities along Common NORMAL!

$$e = \left.\frac{v_{Relative\ Separation}}{v_{Relative\ Approach}}\right|_{Normal} = \frac{v_{Bn}^* - v_{An}^*}{v_{An} - v_{Bn}}$$

(Perfectly Plastic) $0 \le e \le 1$ (Perfectly Elastic)

# Particle Kinetics: Impulse-Momentum/ Impact

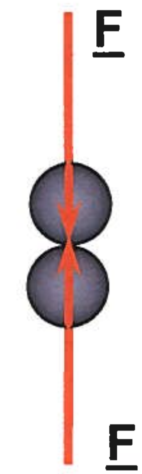



- Solving constraint relations !

$$(1)\quad v_{At} = v_{At}^* \;\;\&\;\; v_{Bt} = v_{Bt}^*$$

$$(2)\quad v_{Bn}^* = v_{Bn} + \frac{m_A}{m_B}\left(v_{An} - v_{An}^*\right)$$

$$(3)\quad v_{Bn}^* = e\left(v_{An} - v_{Bn}\right) + v_{An}^*$$




- From which the unknown rebound (normal) component of velocities become

$$(2)=(3) \Rightarrow (4)\quad v_{An}^* = \left(\frac{m_A - m_B e}{m_A + m_B}\right)v_{An} + \left(\frac{m_B}{m_A + m_B}\right)(1+e)v_{Bn}$$

$$(4) \rightarrow (3) \Rightarrow (5)\quad v_{Bn}^* = \left(\frac{m_A}{m_A + m_B}\right)(1+e)v_{An} + \left(\frac{m_B - m_A e}{m_A + m_B}\right)v_{Bn}$$

$$\underline{\mathbf{V}}_A^* = \left(v_{A_t}, v_{A_n}^*\right) \;\;\&\;\; \underline{\mathbf{V}}_B^* = \left(v_{B_t}, v_{B_n}^*\right)$$

# Particle Kinetics: Impulse-Momentum/ Impact

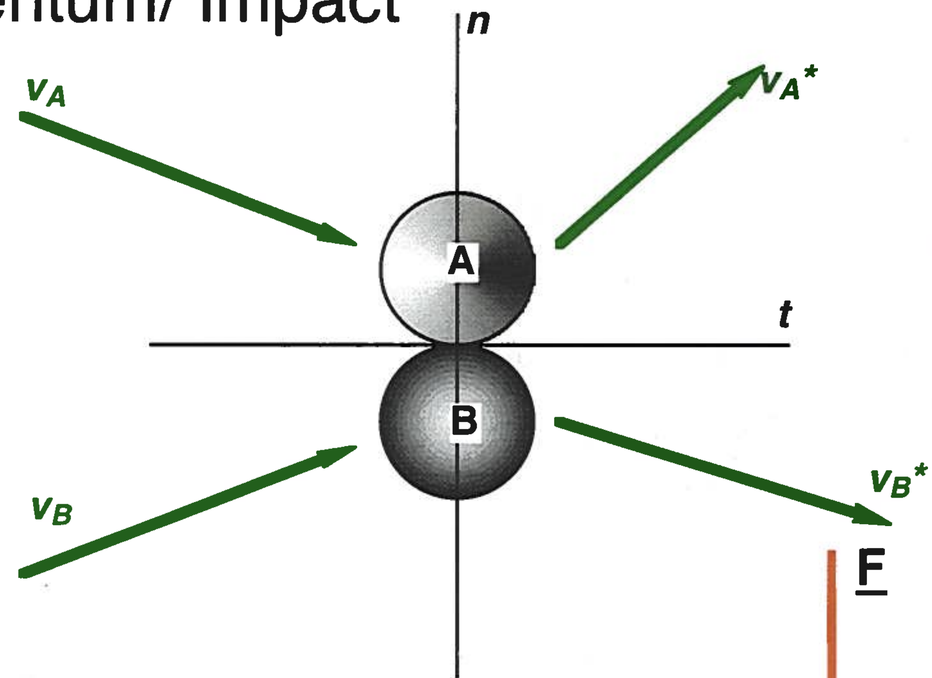
- What if $m_B >>> m_A$ ?

$$(1)\ v_{At} = v^*_{At} \ \ \&\ \ v_{Bt} = v^*_{Bt}$$

$$(2)\ v^*_{Bn} = v_{Bn} + \frac{m_A}{m_B}\left(v_{An} - v^*_{An}\right)$$

0

$$(3)\ v^*_{Bn} = e\left(v_{An} - v_{Bn}\right) + v^*_{An}$$





- From which the unknown rebound (normal) component of velocities become

$$(2) \rightarrow (4)\ v^*_{Bn} = v_{Bn}$$

$$(4) \rightarrow (3) \Rightarrow (5)\ v^*_{An} = -e\,v_{An} + (1+e)\,v_{Bn}$$

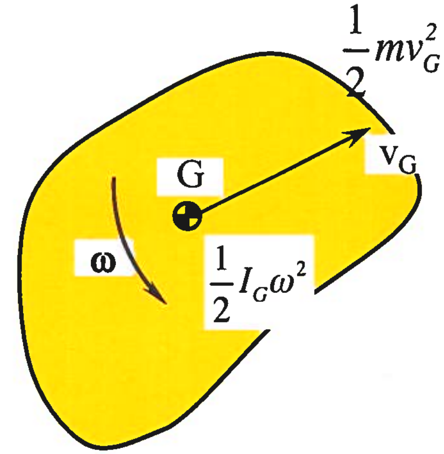
$$\underline{\mathbf{V}}^*_A = \left(v_{A_t}, v^*_{A_n}\right) \ \ \&\ \ \underline{\mathbf{V}}^*_B = \underline{\mathbf{V}}_B$$

## Particle Kinetics: **WORK-ENERGY for Rigid Bodies** (Scalar!)

$$U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}$$

- Need to incorporate the **ROTATION** elements
  - **Kinetic Energy of Rigid Bodies:**

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$

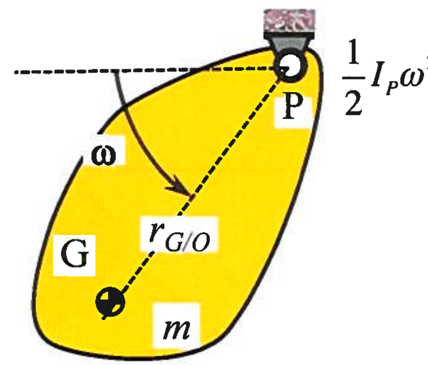



- For fixed axis of rotation P other than **CG**.

$$v_G = r_{G/O}\omega^2$$

$$T = \frac{1}{2} m (r_{G/O}\omega)^2 + \frac{1}{2} I_G \omega^2$$

$$= \frac{1}{2}(I_G + m r_{G/O}^2)\omega^2 = \frac{1}{2} I_P \omega^2$$




- Use either CG or fixed axis of rotation P!!!!

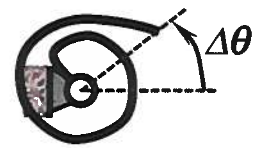
# Particle Kinetics: WORK-ENERGY for Rigid Bodies (Scalar!)

$$U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}$$

- Need to incorporate the **ROTATION** elements
  - **Conservative Forces (now Moments):**
    - Springs (linear & torsional)

$$F_S = -k_s\,(l - l_0)$$ $k_s$ – *stiffness (Force/Length)* $l_0$ – *unstretched length*

$$M_S = -k_\theta\,(\theta - \theta_0)$$ $k_\theta$ – *torsional stiffness (torque/radian)* $\theta_0$ – *unstretched angle*




- Potential Functions

$$\Delta V_e = \Delta V_{e_s} + \Delta V_{e_\theta} = \frac{1}{2}k_s(\Delta l_f^2 - \Delta l_i^2) + \frac{1}{2}k_\theta(\Delta\theta_f^2 - \Delta\theta_i^2)$$

- Constant Torques can also be treated as Potential functions

$$\Delta V_{e_\theta} = M\Delta\theta \qquad \Delta V_{g_y} = W\Delta h = mg\Delta h$$

# Particle Kinetics: WORK-ENERGY for Rigid Bodies (Scalar!)

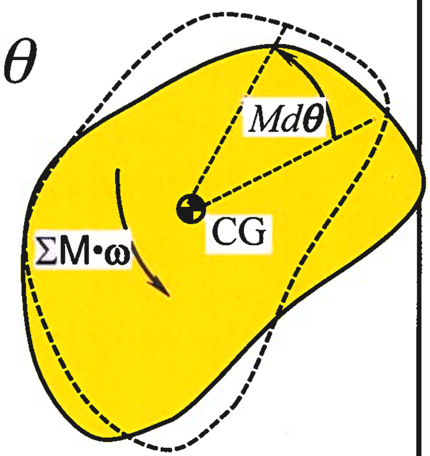
$$U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}$$

- Need to incorporate the **ROTATION** elements
  - **Work:**

    ***FORCE/MOMENT*** applied thru a ***CURVILINEAR/ANGULAR DISPLACEMENT***

    - No displacement -- NO WORK!

$$U_{1-2} = \int_{r_1}^{r_2} \underline{\mathbf{F}} \bullet d\underline{\mathbf{r}} + \int_{\theta_1}^{\theta_2} \underline{\mathbf{M}} \bullet d\underline{\boldsymbol{\theta}} = \int_{s_1}^{s_2} F_t\, ds + \int_{\theta_1}^{\theta_2} M\, d\theta$$




    - Units ENERGY: SI: Joules (1 N-m) FPS: ($lb_f$-ft)
  - **Power** : work/time

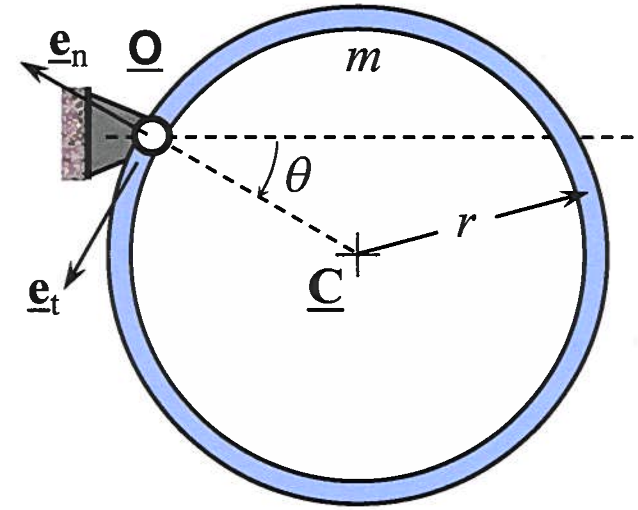
$$P = dU/dt = \underline{\mathbf{F}} \bullet \underline{\mathbf{v}} + \underline{\mathbf{M}} \bullet \underline{\boldsymbol{\omega}}$$

    - Units SI: Watt (Joules/sec) FPS: 1 Horsepower = 550 ft-#/sec

# Rigid Body Kinetics – Planar Motion (2D)

**Revisit from last class:**

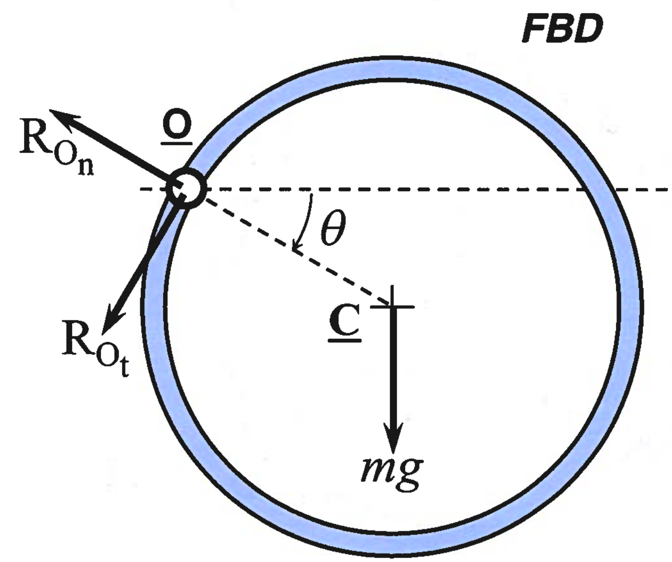
- A thin ring of mass $m$ is free to rotate in the vertical plane about the frictionless pin joint at ***O***.
- Its angular velocity is $\omega_0$ (CW) when $\theta$=0°




**Find:** *(for any arbitrary angle θ)*

- The angular velocity of the ring
- ~~The reactions forces at **O**~~

**Solution:**



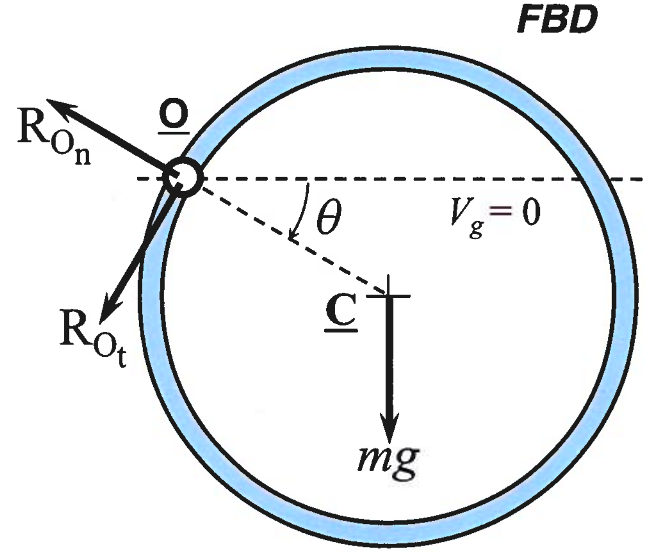

- Fixed axis rotation at frictionless pin joint **O**
- FBD constructed using $n$-$t$ axes (+z into page – CW +)
- Forces, displacements, velocities => W-E!
  - *Last time Integrated ΣM to get ω=f(θ)*

$$U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}$$

$\Rightarrow \Delta E_{TOT} = 0$ !!!!

# Rigid Body Kinetics – Planar Motion (2D)

**W-E Solution (continued):**




$$I_C = mr^2$$

$$I_O = 2mr^2$$

$$\Delta E_{TOT} = \Delta T + \Delta V_g = 0$$

$$\Delta T = \frac{1}{2} I_O (\omega_\theta^2 - \omega_0^2)$$

$$\Delta V_g = W \Delta h = mg \Delta h = mg(-r \sin\theta)$$

$$\Delta E_{TOT} = \frac{1}{2} I_0 (\omega_\theta^2 - \omega_0^2) - mgr \sin\theta = 0$$

$$\omega_\theta^2 = \omega_0^2 + \frac{2mgr \sin\theta}{I_0} = \omega_0^2 + \frac{2mgr \sin\theta}{2mr^2}$$

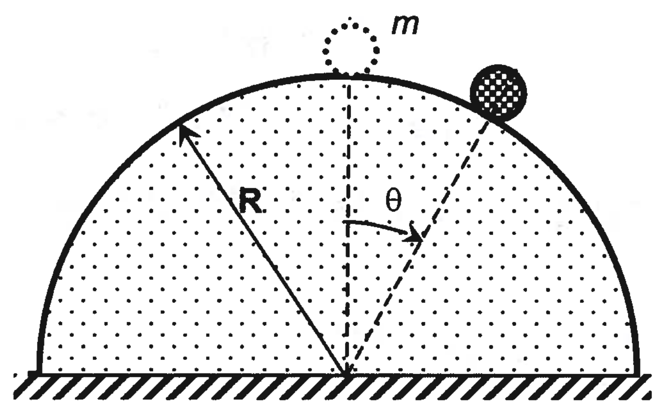
$$\omega_\theta = \sqrt{\omega_0^2 + \frac{g}{r} \sin\theta}$$

- Reaction forces? See earlier example using Newton's Laws

# Conservation-Energy Example ref Bedford & Fowler 15.85

**Given:**

- A small pellet of mass *m* and neglible diameter, sits atop a smooth circular cylinder of radius R.
- The pellet is given a slight nudge

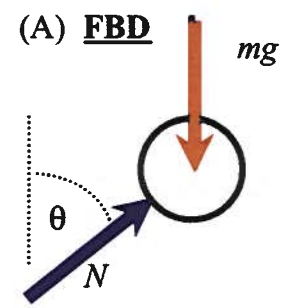



**Find:**

(A) Draw a correct FBD for the pellet in general position θ

(B) The value of θ where pellet loses contact with the cylinder

(C) The pellet's speed at the point where it loses contact




**Solution:**

- FBD of pellet in general position (working over a motion interval here)
- Identify
  - Conservative Forces — **mg** (Weight/Gravity)
  - Non-working Constraint Forces — **N** (Cylinder reaction force)

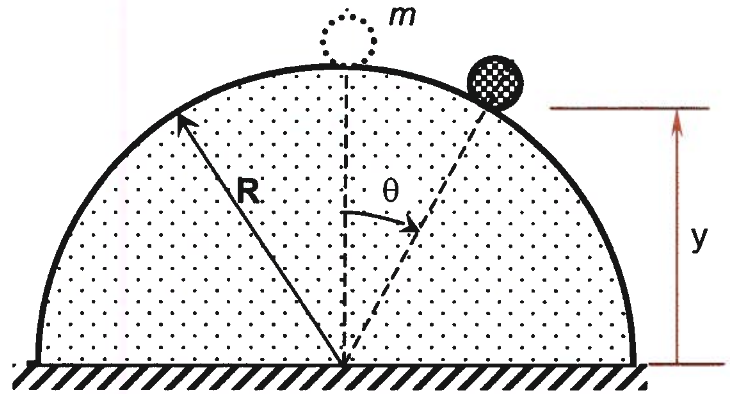
# Conservation-Energy Example ref Bedford & Fowler 15.85

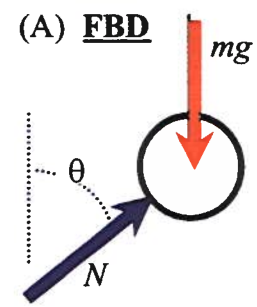
Solution:

- ALL Forces are either Conservative or Non-working constraints, therefore Cons. Of Energy applies!

$$\Delta E_{sys} = \Delta T + \Delta V_g = 0$$

- It starts from rest, $v_0$=0 @ $\theta_0$=0
- Set the datum for potential @ base of the cylinder (y=Rcosθ)




$$\Delta E_{sys} = 0 = \Delta T + \Delta V_g$$
$$0 = 1/2\, m\left(v_\theta^2 - 0\right) + mg\left(R\cos\theta - R\right)$$
$$v_\theta^2 = 2gR(1-\cos\theta)$$

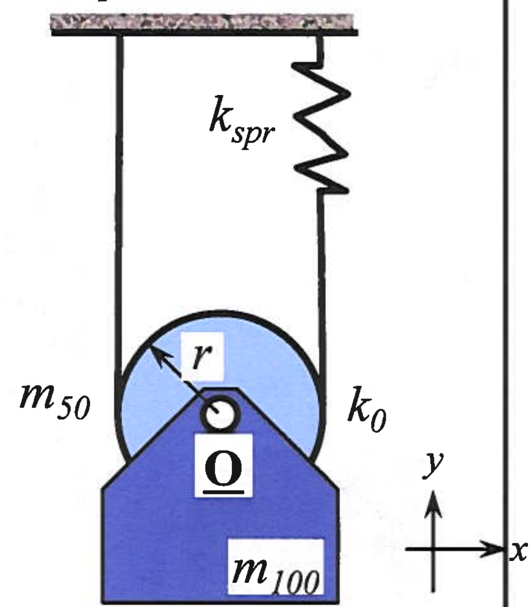
- Just as the pellet loses contact (N=0)

$$\sum F_n = ma_n$$
$$mg\cos\theta = m\frac{v_\theta^2}{R}$$
$$v_\theta^2 = Rg\cos\theta$$

- Equating expressions for $v_\theta^2$ yields

$$v_\theta^2 \Rightarrow Rg\cos\theta = 2gR(1-\cos\theta)$$
$$3\cos\theta = 2$$
$$\theta = \cos^{-1}\left(2/3\right) \cong 48^\circ$$

# Rigid Body Kinetics – Planar Motion (2D)




## Given:

- A rotating sheave ($m_{50}$) carries a high strength, electromagnet ($m_{100}$)

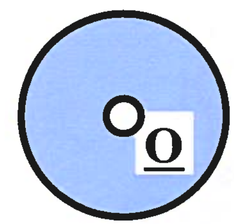
  $r=0.4m \quad k_0=0.3\ m \quad m_{50}=50\ kg \quad m_{100}=100\ kg$

- Released from rest with the spring initially stretched 0.1 m $\quad k_{spr}=1.5\ kN/m$

## Find:

- Velocity of **O** after it has dropped $\Delta y_O=0.5\ m$

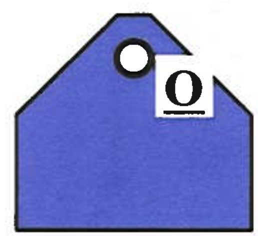
FBD




## Solution:

- 2 RB, CG's motion rectilinear + sheave rotation
- Set coordinate $x$-$y$ axes horiz/vert with CCW+
- BC's loads, displacements, velocities => W-E!

$$U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}$$




- Finish FBD & see if system is conservative!

# Rigid Body Kinetics – Planar Motion (2D)

**W-E Solution (continued):**

- Conservative loads: $F_{spr},\ m_{50}g,\ m_{100}g$
- Forces DO NO WORK: $T$ => *displacement = 0!*
- $R_{O_y}$ Internal reaction not requested – System?

$$\Delta E_{sys} = \Delta T + \Delta V_g + \Delta V_e = 0 \quad !!$$

$$\Delta T_{sys} = \frac{1}{2} m_{100}(v_O^2 - 0) + \frac{1}{2} m_{50}(v_O^2 - 0) + \frac{1}{2} I_O(\omega^2 - 0)$$

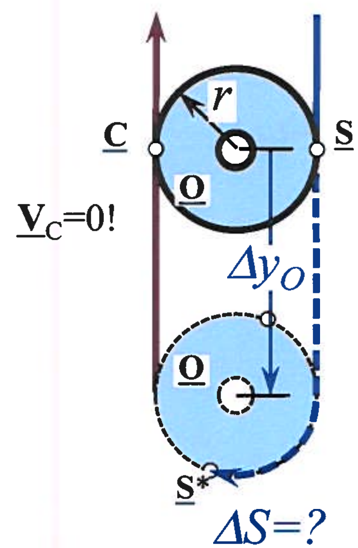
$$= \tfrac{1}{2}(m_{100} + m_{50})v_O^2 + \tfrac{1}{2} m_{50} k_O^2 \omega^2$$

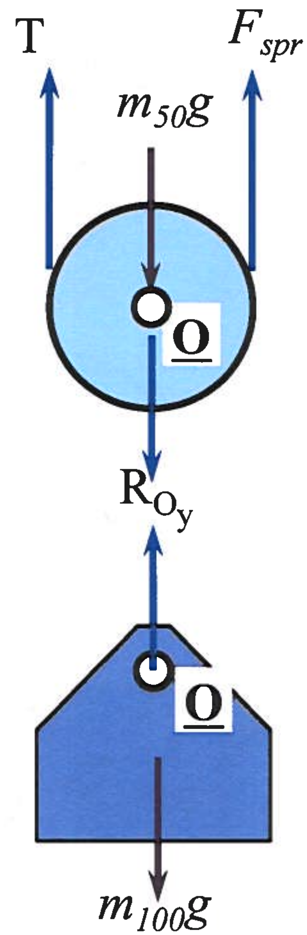
$$\Delta V_g = W\Delta h = (m_{50} + m_{100})g\Delta y_O$$

$$\Delta V_e = \frac{1}{2} k_{spr}\{(\Delta S + S_O)^2 - S_0^2\}$$

$$\underline{\mathbf{V}}_O = \underline{\mathbf{V}}_C + \omega \underline{\mathbf{k}} \times r\underline{\mathbf{i}} \quad => \quad v_O = \omega\, r$$

$$=> \quad v_S = \omega(2r) = 2v_O$$




$$\Delta y_O = r\Delta\theta$$

$$\Delta S = 2\Delta y$$

# Rigid Body Kinetics – Planar Motion (2D)

**W-E Solution (continued):**

- Assembling the terms $\Delta E_{sys} = \Delta T + \Delta V_g + \Delta V_e = 0$

$$\Delta E_{TOT} = 0 = \begin{bmatrix} \tfrac{1}{2}\{m_{100}r^2 + m_{50}(r^2 + k_O^2)\}\omega^2 \\ +(m_{50} + m_{100})g\Delta y_O + \frac{1}{2}k_{spr}\{(2\Delta y_O + S_O)^2 - S_0^2\} \end{bmatrix}$$

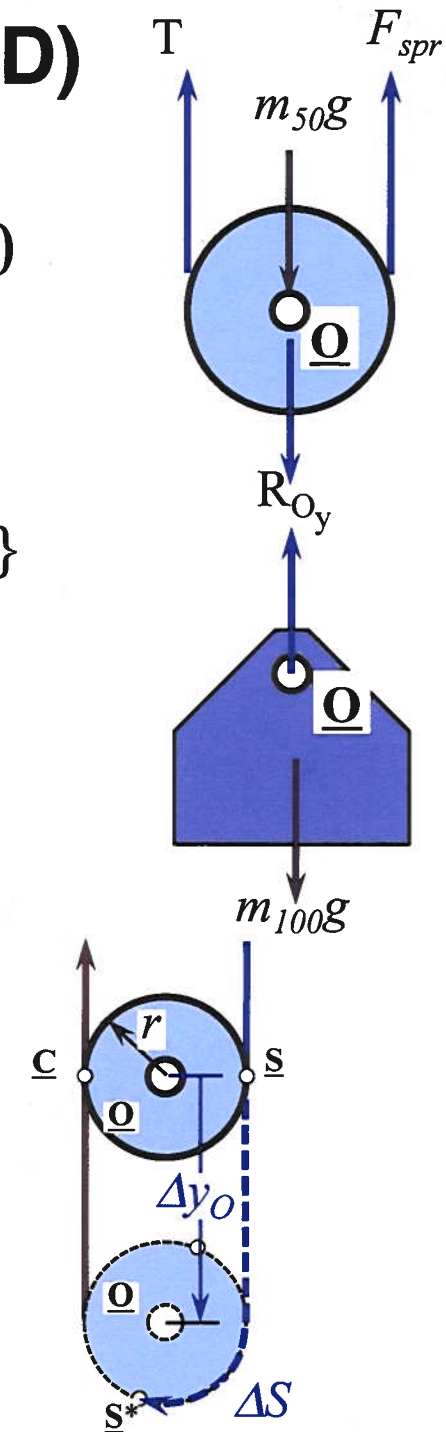
$$\omega = \sqrt{\frac{2(m_{50} + m_{100})g\Delta y_O + k_{spr}\{(2\Delta y_O + S_O)^2 - S_0^2\}}{m_{100}r^2 + m_{50}(r^2 + k_O^2)}}$$

$r=0.4m$ $k_{spr}=1.5\ kN/m$ $m_{100}=100\ kg$

$\Delta y=-0.1m$ $k_0=0.3\ m$ $S_0=0.1\ m$ $m_{50}=50\ kg$

$\omega = 3.5r/s\ \ CW$

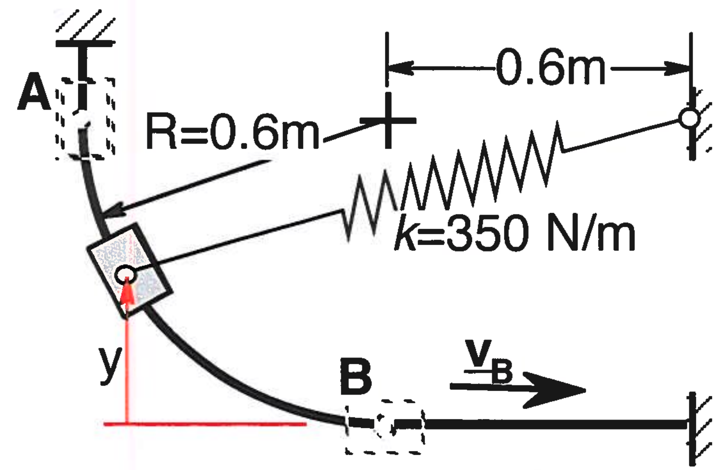
$=> v_O = r\omega = 0.4m * 3.5r/s = \underline{\underline{14\ m/s\ (-\underline{j})}}$



# Conservation-Energy Example ref ~Meriam & Kraige 3/17

Given:

- $m$ =3 kg slider on circular track shown
- Starting from A with $v_A$=0
- $l_o$=0.6 m (unstretched), $k$=350 N/m
- $\mu$=0 (i.e friction is negligible)

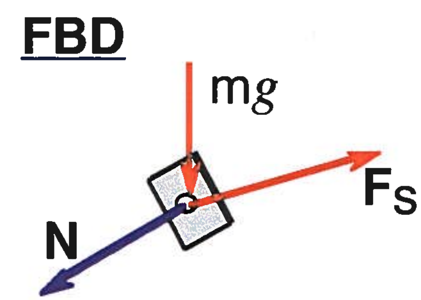



Find:

(A) Velocity of slider as it passes **B**

Solution:

- FBD of crate in general position (working over a motion interval here)




- Identify
  - Conservative Forces

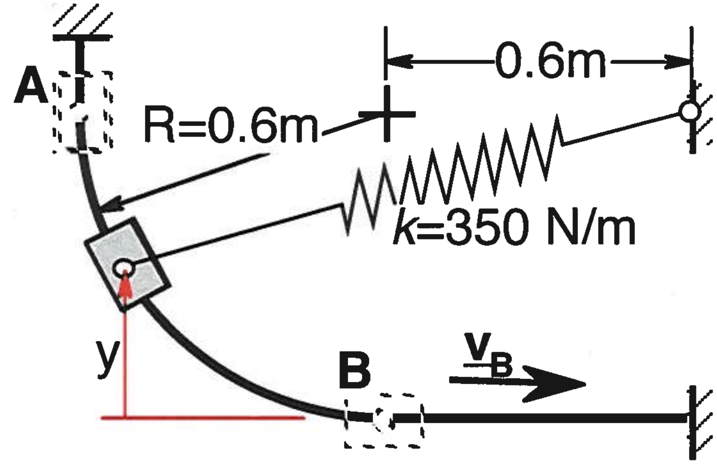
    **mg** (Weight/Gravity) & **$F_s$** (Spring)
  - Non-working Constraint Forces

    **N** (Track reaction force)

# Conservation-Energy Example ref ~Meriam & Kraige 3/17

Solution:

- ALL Forces are either Conservative or Non-working constraints, therefore Cons. Of Energy applies!



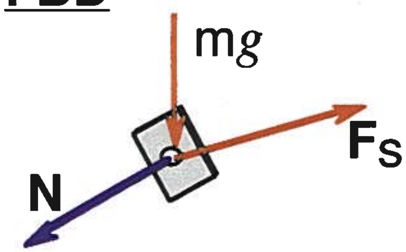

$$\Delta E_{TOT} = \Delta T + \Delta V_g + \Delta V_e = 0$$

$$\Delta T_{AB} = \frac{1}{2}m(v_B^2 - v_A^2) = \frac{1}{2}m(v_B^2 - 0)$$

$$\Delta V_{AB\,g} = mg(y_B - y_A) = mg(0 - R)$$

$$\Delta V_{AB\,e} = \frac{1}{2}k\left\{(l_B - l_0)^2 - (l_A - l_0)^2\right\}$$

**FBD**



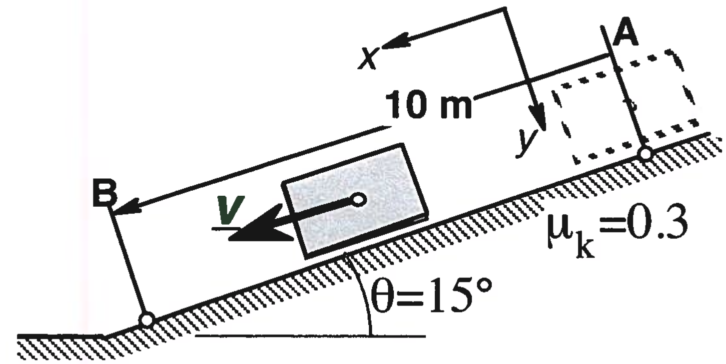

- Pulling together all components & isolating $v_B$

$$v_B = \sqrt{2gR + \frac{k}{m}\left\{R^2 - (\sqrt{2}R - R)^2\right\}}$$

- Incorporating numerical values of all terms

$$v_B = \sqrt{2*9.81(m/s^2)(0.6m) + \frac{350N/m}{3kg}\left\{(0.6m)^2 - (\sqrt{2}*0.6m - 0.6m)^2\right\}} = \underline{\underline{6.82\ \ m/s}}$$

# Work-Energy Example ref ~Meriam & Kraige 3/11

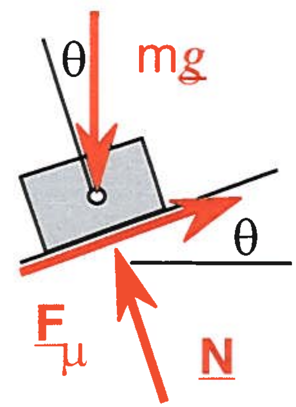



Given:

- A crate of mass $m$ slides down an incline
- $m$ =50 kg, θ=15°, $\mu_k$=0.3,
- Reaches **A** with speed 4 m/s

Find:

(A) Speed of crate $v_B$ as it reaches a point **B** 10 m down the incline from **A**

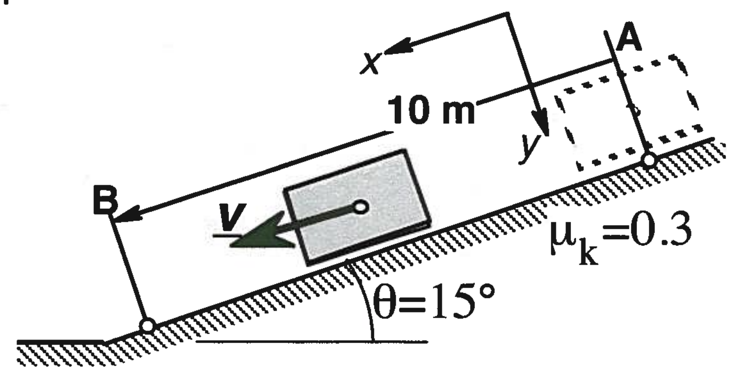



Solution:

- Rectilinear motion, align axes accordingly -i.e. **II** & ⊥ to incline
- FBD of crate in general position (working over a motion interval here)
- No movement ⊥ to incline so Newtons Law says -?

$$\sum F_y = mg\cos\theta - N = 0 \quad \Rightarrow \quad N = mg\cos\theta$$

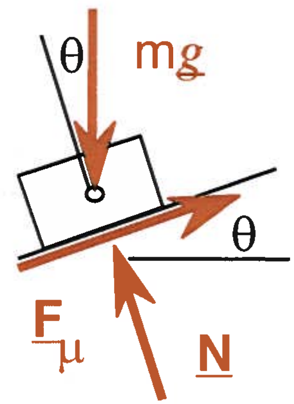
# Work-Energy Example ref ~Meriam & Kraige 3/11




Solution (cont'd):

- Work done is due to the resultant forces *in direction of displacement* (i.e. down incline) & includes Friction & component of Weight

$$U_{A\text{-}B} = (mg\ \sin\theta - N\mu_k)\Delta x_{AB}$$
$$= (mg\ \sin\theta - mg\ \cos\theta\mu_k)\Delta x_{AB}$$




- Principle of Work-Energy then says

$$U_{A\text{-}B} = \Delta T_{A\text{-}B} = T_B - T_A$$

$$\Rightarrow \quad T_B = U_{A\text{-}B} + T_A$$

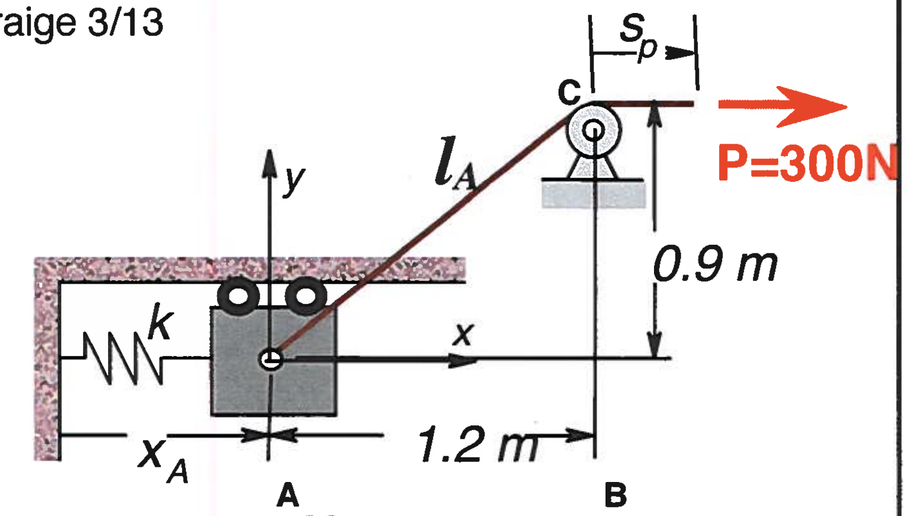
$$\frac{1}{2}mv_B^2 = mg(\sin\theta - \cos\theta\mu_k)\Delta x_{AB} + \frac{1}{2}mv_A^2$$

$$v_B = \sqrt{2g(\sin\theta - \cos\theta\mu_k)\Delta x_{AB} + v_A^2}$$

$$v_B = \sqrt{2 * 9.81(m/s^2) * (\sin 15° - \cos 15° * 0.3) * 10m + (4m/s)^2}$$

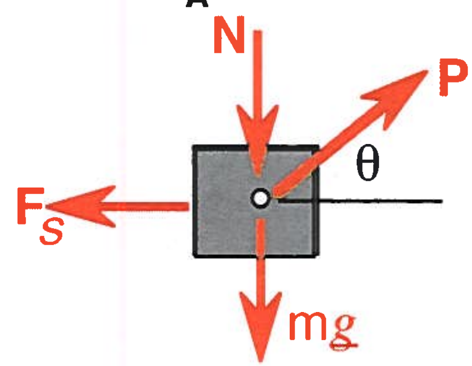
$$v_B = \underline{\underline{3.15\ \ m/s}}$$

# Work-Energy: Example ref ~Meriam & Kraige 3/13




Given:

- Block ($m$ =50 kg) mounted on rollers
- Massless spring w/ $k$=80 N/m
- Released from rest at **A** where spring has initial stretch of 0.233 m
- Cord w/ constant tension <u>**P**</u>=300 N attaches to block & routed over frictionless/massless (ideal) pulley @ **C**

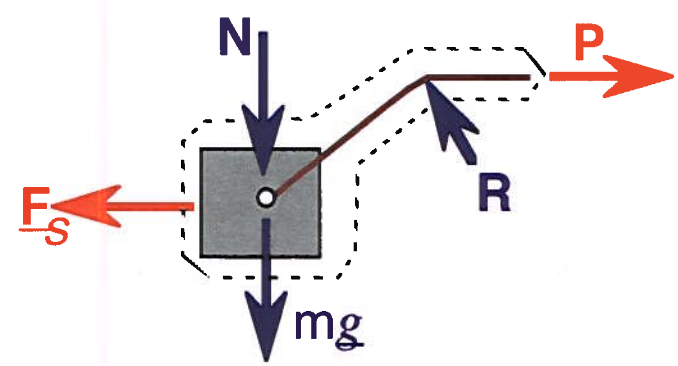



Find:

(A) Speed of block $v_B$ as it reaches a point **B** directly under the pulley.

Solution:

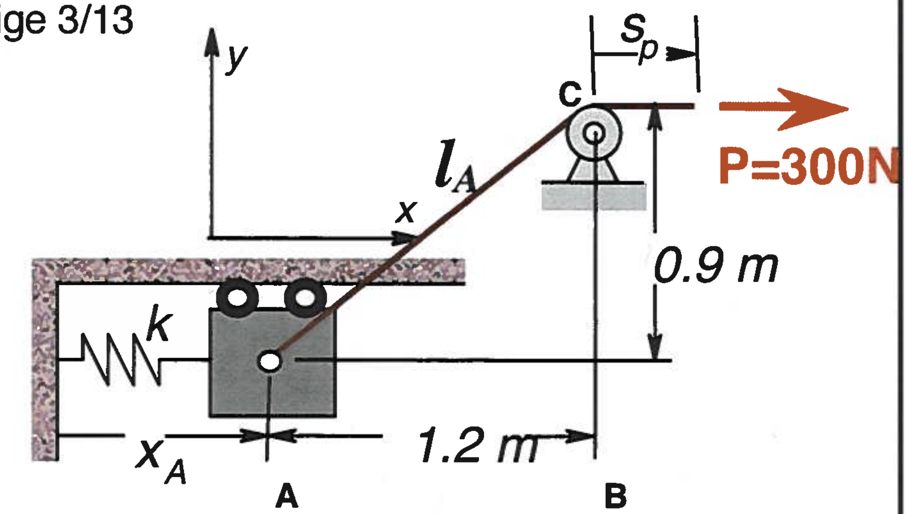
- Again, rectilinear motion, align axes accordingly
- FBD of block in general position (working over a motion interval here)




- Look at alternative - include the rope in as part of the SYSTEM - reduce FDB to an ***<u>ACTIVE Force Diagram!</u>***

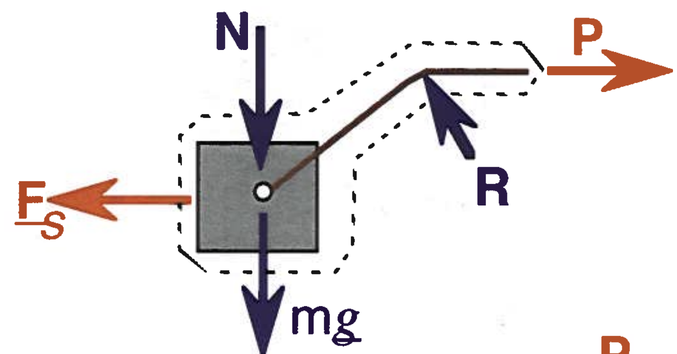
# Work-Energy: Example ref ~Meriam & Kraige 3/13

Solution (cont'd):




***ACTIVE Force Diagram!***

- Eliminate Normal Forces ⊥ to displacement @ their point of contact {THEY DO NO WORK!}
  - Weight (**mg**) & Roller reactions (**N**)
  - Pulley force on rope (**R**)

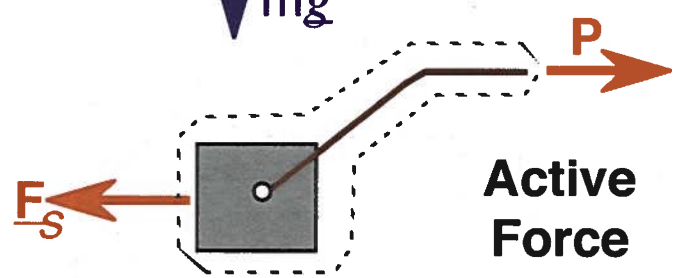



- Active forces DO work on the system
  - Spring Force ($F_s$) => opposes motion

$$F_s = -kx$$

$$U_{AB_{Fs}} = \int_{x_A}^{x_B} F_s\,dx = \int_{x_A}^{x_B} -kx\,dx$$

$$= -\frac{1}{2}kx^2\Big|_{x_A}^{x_B} = -\frac{1}{2}k(x_B^2 - x_A^2)$$




**Active Force Diagram**

  - Assuming block can actually reach B

$$U_{AB_{Fs}} = -\frac{1}{2}80(N/m)\{(1.2+0.233)^2 - 0.233^2\}(m^2) = -80\,Joules$$

# Work-Energy: Example ref ~Meriam & Kraige 3/13

Solution (cont'd):

- Calculate Work done on system by P
  - Cord Tension (P) => *constant*
  - Displacement of P

$$L_{cord} = s_P + l = constant$$

$$\Delta s_P = -\Delta l = l_A - l_B$$

$$= \sqrt{1.2^2 + 0.9^2} - 0.9 \cong 0.61m$$

$$U_{AB_P} = P\Delta s = 300(N) * 0.61(m)$$

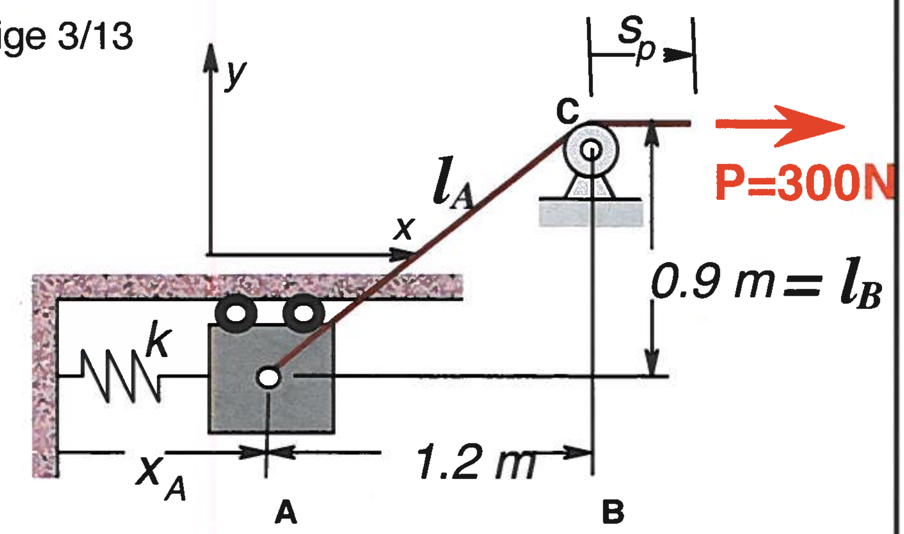
$$= 180\,Joules$$

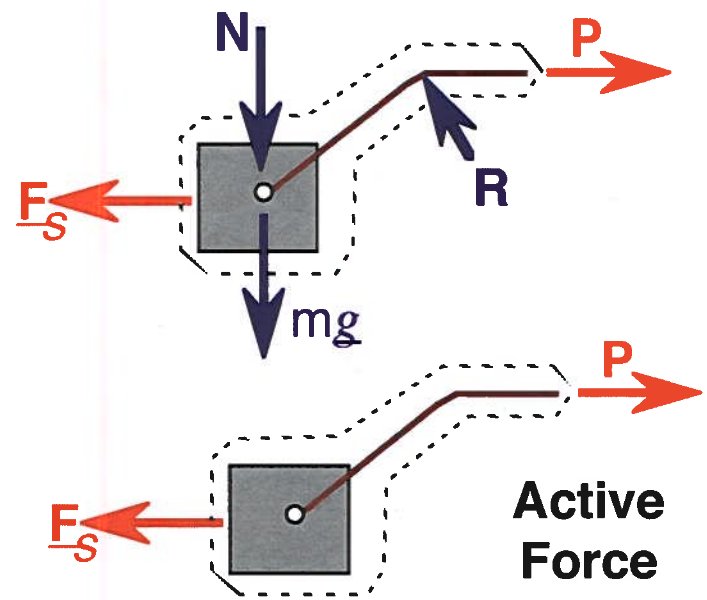
- Work-Energy

$$\mathrm{U}_{\mathrm{TOT}} = \Delta \mathrm{T}_{\mathrm{A\text{-}B}} = \mathrm{T}_{\mathrm{B}} - \mathrm{T}_{\mathrm{A}}$$

$$-80 + 180(Joules) = \frac{1}{2} m(v_B^2 - 0)$$

$$\Rightarrow v_B = \sqrt{\frac{100(Joules) * 2}{50\,Kg}} = \underline{\underline{2.0\,m/s}}$$

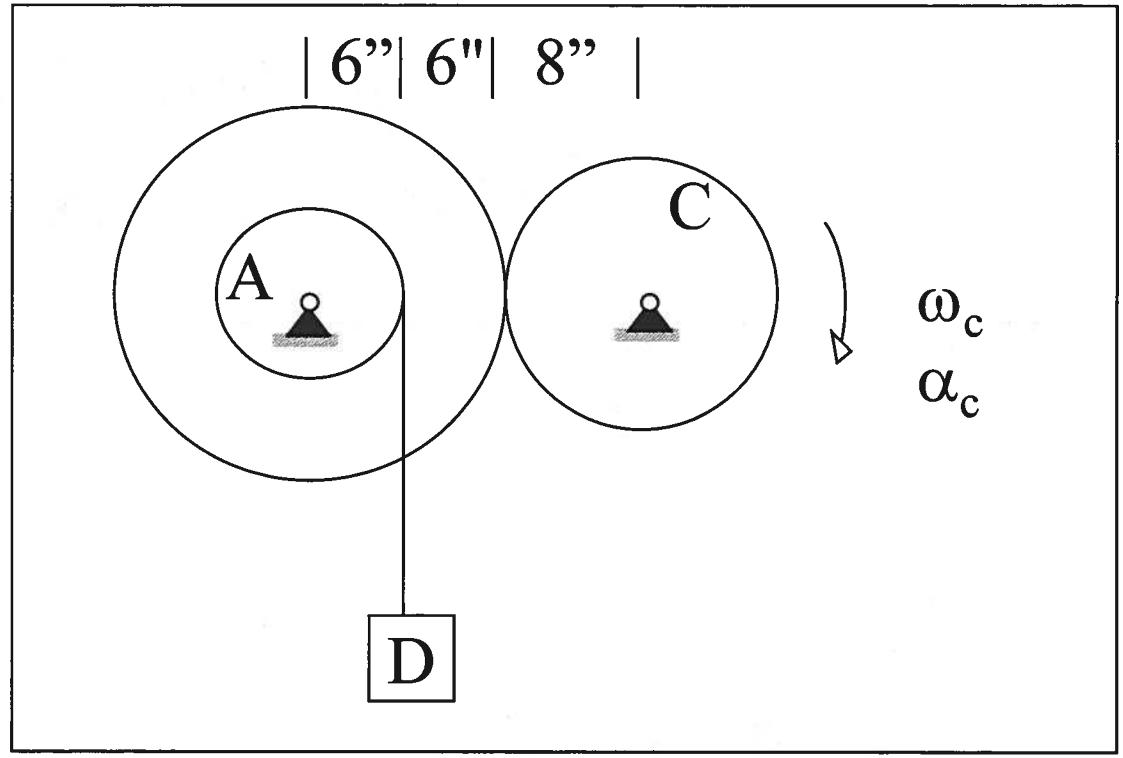



Active Force Diagram

**Given:** $\omega_c = 2$ r/s (clockwise)

$\alpha_c = 6$ r/s$^2$ (clockwise)




**Find:** $v_D$, $a_D$

$r_A = 6"$  $r_B = 12"$  $r_C = 8"$

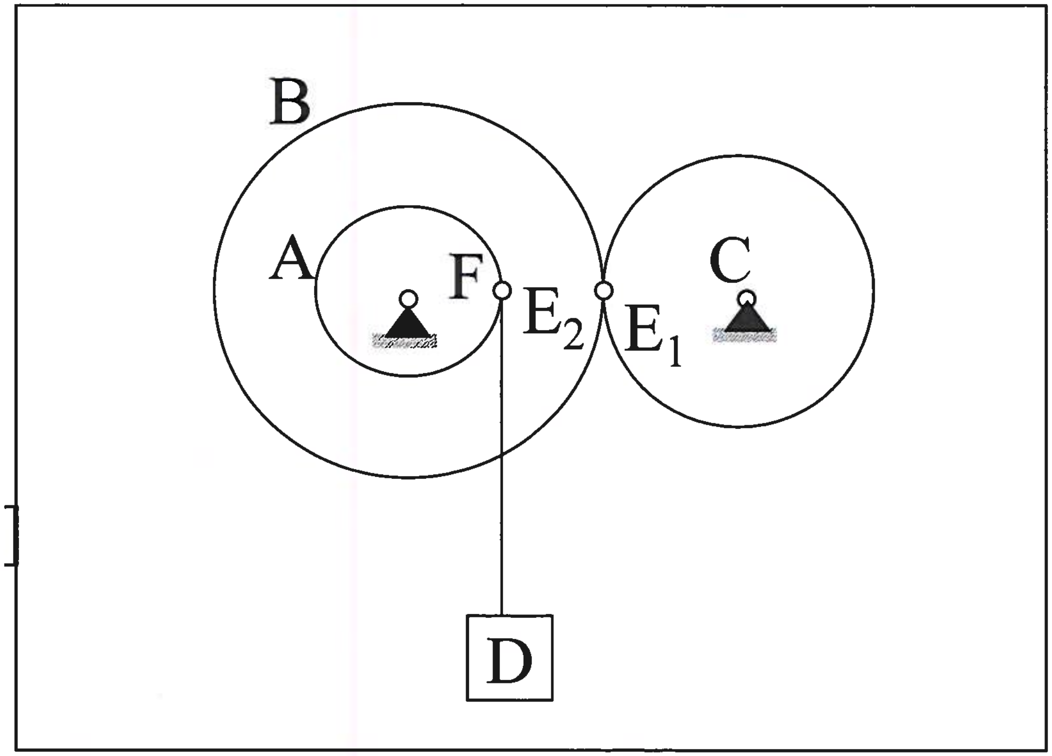
$\omega_C = 2 \text{ r/s} \circlearrowright$  $\alpha_C = 6 \text{ r/s}^2 \circlearrowright$

$V_{E1} = r_C\omega_C = 8 \cdot 2 = 16 \text{ in/s} \uparrow$

$V_{E2} = 16 \uparrow = r_B\omega_B = 12\,\omega_B$

so: $\omega_B = 4/3 \circlearrowleft = \omega_A$ [r/s]

$V_D = V_F = \omega_A r_A = 4/3\,(6) = 8 \uparrow$ [in/s]



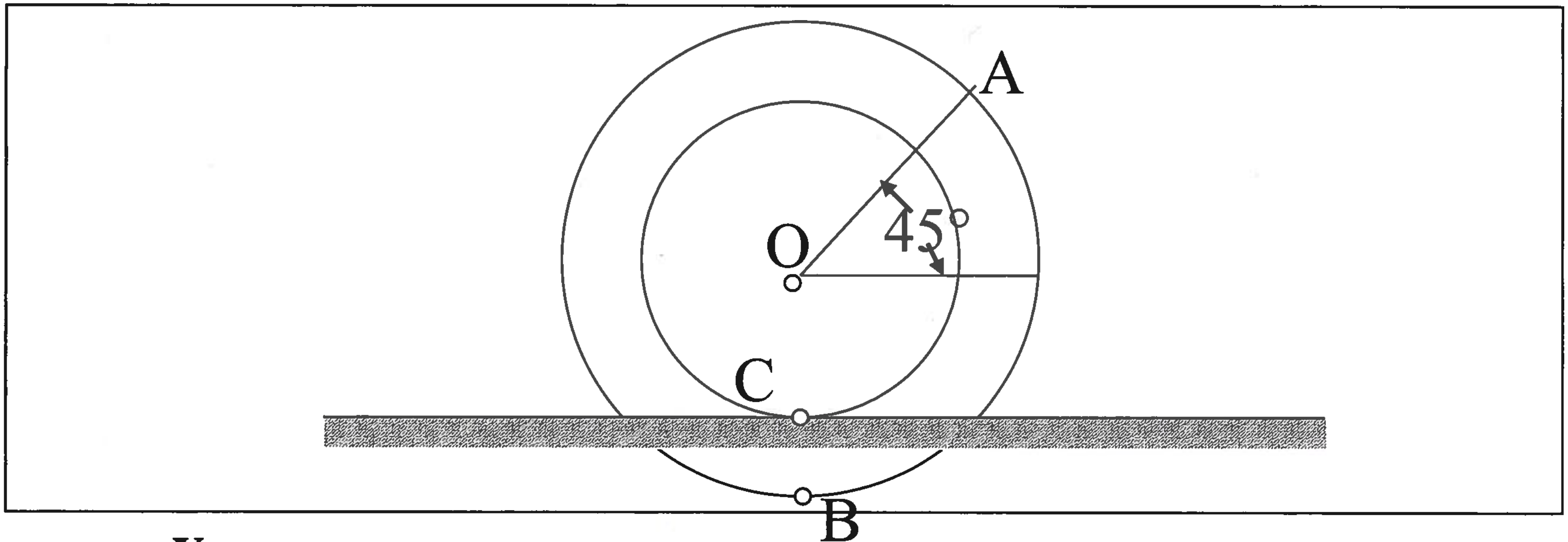

$a_{E1} = \alpha_C r_C \uparrow + \omega_C^2 r_C \rightarrow$

$= 6\,(8) \uparrow + 4(8) \rightarrow = 48 \uparrow + 32 \rightarrow$ [in/s²]

$a_{E2\,t} = a_{E1t} = 48 \uparrow = \alpha_B r_B = \alpha_B(12)\ \alpha_B = 4 \circlearrowleft = \alpha_A$

$a_{F\,t} = \alpha_A r_A = 4(6) = 24 \uparrow = a_D$ [in/s²]

**Given:** $r_o = 3'$ $r_i = 2'$ $v_o = 10$ f/s no slip




**Find:** $v_B$

$\underline{v}_o = 10$ ft/s ➡

$\underline{v}_A = ?$

$v_c = v_o + \omega r = 10 - 2\omega = 0$

$\omega = 5$ ↻ or $-5\,\underline{k}$

$\underline{v}_B = \underline{v}_o + \underline{\omega} \times \underline{r}_{B/o} = 10\,\underline{i} + -5\underline{k} \times -3\underline{j} = -5\,\underline{i}$ [ft/s] ⬅

or $\underline{v}_B = \underline{v}_c + \underline{\omega} \times \underline{r}_{B/c} = 0 + 5\underline{k} \times -1\underline{j} = -5\,\underline{i}$ [ft/s] ⟵